130. Ceva's theorem.

Theorem. ***Given a triangle*** ABC (Figure 138) ***and points*** A', B', ***and*** C' ***on the sides*** BC, CA, ***and*** AB ***respectively, the lines*** AA', BB', ***and*** CC' ***are concurrent if and only if the vertices can be equipped with masses such that*** A', B', C' ***become centers of mass of the pairs:*** B ***and*** C, C ***and*** A, A ***and*** B ***respectively.***

Suppose A, B, and C are material points, and A', B' and C' are positions of the centers of mass of the pairs B and C, C and A, A and B. Then, by the regrouping property, the center of mass of the whole system lies on each of the segments AA', BB', and CC'. Therefore these segments are concurrent.

Conversely, assume that the lines AA', BB', and CC' are concurrent. Assign an arbitrary mass $m_A = m$ to the vertex A, and then assign masses to the vertices B and C so that C' and B' become the centers of mass of the pairs A and B, and A and C respectively, namely:

$$m_B = \frac{AC'}{C'B}m, \quad \text{and} \quad m_C = \frac{AB'}{B'C}m.$$

Then the center of mass of the whole system will lie at the intersection point M of the segments BB' and CC'. On the other hand, by regrouping, it must lie on the line connecting the vertex A with the center of mass of the pair B and C. Therefore the center of mass of this pair is located at the intersection point A' of the line AM with the side BC.

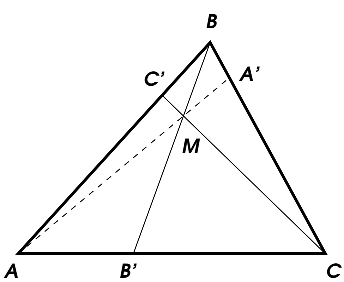

Figure 138

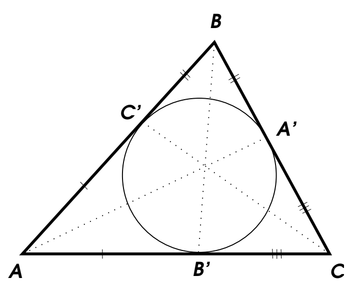

Figure 139

Corollary (Ceva's theorem). *In a triangle* ABC, *the segments* AA', BB', *and* CC', *connecting the vertices with points on the opposite sides, are concurrent if and only if*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1. \qquad (*)$$

Indeed, when the lines are concurrent, the equality becomes obvious when rewritten in terms of the masses:

$$\frac{m_B}{m_A} \cdot \frac{m_C}{m_B} \cdot \frac{m_A}{m_C} = 1.$$

Conversely, the relation $(*)$ means that if one assigns masses as in the proof of the theorem, i.e. so that $m_B : m_A = AC' : C'B$ and $m_C : m_A = AB' : B'C$, then the proportion $m_C : m_B = BA' : A'C$ holds too. Therefore all three points C', B', and A' are the centers of mass of the corresponding pairs of vertices. Now the concurrency property is guaranteed by the theorem.

Problem. *In a triangle ABC* (Figure 139), *let A', B', and C' denote points of tangency of the inscribed circle with the sides. Prove that the lines AA', BB', and CC' are concurrent.*

Solution 1. We have: $AB' = AC'$, $BC' = BA'$, and $CA' = CB'$ (as tangent segments drawn from the vertices to the same circle). Therefore the relation $(*)$ holds true, and the concurrency follows from the corollary.

Solution 2. Assigning masses $m_A = 1/AB' = 1/AC'$, $m_B = 1/BC' = 1/BA'$, and $m_C = 1/CA' = 1/CB'$, we make A', B', and C' the centers of mass of the corresponding pairs of vertices, and therefore the concurrency follows from the theorem.

131. Menelaus' theorem.

Lemma. ***Three points** A_1**,** A_2**, and** A_3 **are** **collinear** (i.e. lie on the same line) ***if and only if they can be equipped with non-zero pseudo-masses*** m_1**,** m_2**, and** m_3 (they are allowed therefore to have different signs) ***such that***

$$m_1 + m_2 + m_3 = 0, \quad \text{and} \quad m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2} + m_3\overrightarrow{OA_3} = \vec{0}.$$

If the points are collinear, then one can make the middle one (let it be called A_3) the center of mass of the points A_1 and A_2 by assigning their masses according to the proportion $m_2 : m_1 = A_1A_3 : A_3A_2$. Then, for any origin O, we have: $m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2} - (m_1+m_2)\overrightarrow{OA_3} = \vec{0}$, i.e. it suffices to put $m_3 = -m_1 - m_2$.

Conversely, if the required pseudo-masses exist, one may assume (changing, if necessary, the signs of all three) that one of them (say, m_3) is negative while the other two are positive. Then $m_3 = -m_1 - m_2$, and the relation $m_1\overrightarrow{OA_1} + m_2\overrightarrow{OA_2} - (m_1 + m_2)\overrightarrow{OA_3} = \vec{0}$ means that A_3 is the position of the center of mass of the pair of material points A_1 and A_2. Thus A_3 lies on the segment A_1A_2.

`Corollary` (Menelaus' theorem.) *Any points A', B', and C'* (Figure 140) *lying on the sides BC, CA, and AB respectively of $\triangle ABC$, or on their extensions, are collinear, if and only if*

$$\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=1.$$

`Remark.` This relation looks identical to $(*)$, and it may seem puzzling how the same relation can characterize triples of points A', B', C' satisfying two different geometric conditions. In fact in Menelaus' theorem (see Figure 140), either one or all three of the points must lie on extensions of the sides, so that the same relation is applied to two mutually exclusive geometric situations. Furthermore, let us identify the sides of $\triangle ABC$ with number lines by directing them as shown on Figure 140, i.e. the side AB from A to B, BC from B to C, and CA from C to A. Then the segments AC', $C'B$, BA', etc. in the above relation can be understood as signed quantities, i.e. real numbers whose absolute values are equal to the lengths of the segments, and the signs are determined by the directions of the vectors $\overrightarrow{AC'}$, $\overrightarrow{C'B}$, $\overrightarrow{BA'}$, etc. on the respective number lines. With this convention, the correct form of the relation in Menelaus' theorem is:

$$\frac{AC'}{C'B}\cdot\frac{BA'}{A'C}\cdot\frac{CB'}{B'A}=-1, \qquad (**)$$

thereby differing from the relation in Ceva's theorem by the sign.[3]

To prove Menelaus' theorem in this improved formulation, note that we can always assign to the vertices A, B, and C some real numbers a, b, and c so that C' (resp. B') becomes the center of mass of the pair of points A and B (resp. C and A) equipped with pseudo-masses a and $-b$ (resp. c and $-a$). Namely, it suffices to take: $-b:a=AC':C'B$ and $-a:c=CB':B'A$. Then the relation $(**)$ means that $BA':A'C=-c:b$, i.e. A' is the center of mass of the pair B and C equipped with pseudo-masses b and $-c$ respectively. Thus, we have: $(a-b)\overrightarrow{OC'}=a\overrightarrow{OA}-b\overrightarrow{OB}$, $(c-a)\overrightarrow{OB'}=c\overrightarrow{OC}-a\overrightarrow{OA}$, and $(b-c)\overrightarrow{OA'}=b\overrightarrow{OB}-c\overrightarrow{OC}$. Adding these equalities, and putting $m_A=b-c$, $m_B=c-a$, $m_C=a-b$, we find:

$$m_A\overrightarrow{OA'}+m_B\overrightarrow{OB'}+m_C\overrightarrow{OC'}=\vec{0},\quad m_A+m_B+m_C=0.$$

[3]In Ceva's theorem, it is also possible to apply the sign convention and consider points on the extensions of the sides. Then the relation $(*)$ remains the correct criterion for the three lines to be concurrent (or parallel). When $(*)$ holds, an even number (i.e. 0 or 2) of the points lie on the extensions of the sides.

Therefore the points A', B', and C' are collinear.

Conversely, for any points C' and B' in the interior or on the extensions of the sides AB and CA, we can find a point A' on the line BC such that the relation $(**)$ holds true. Then, according to the previous argument, points A', B', and C' are collinear, i.e. point A' must coincide with the point of intersection of the lines $B'C'$ and BC. Thus the relation $(**)$ holds true for any three collinear points on the sides of a triangle or on their extensions.

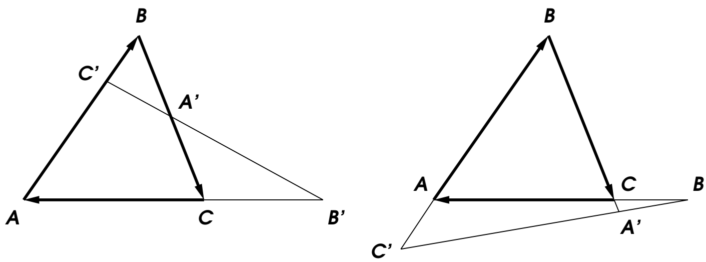

Figure 140

132. The method of barycenters demystified. This method, developed and applied in §§128–131 to some problems of *plane* geometry, can be explained using geometry of vectors in *space*.

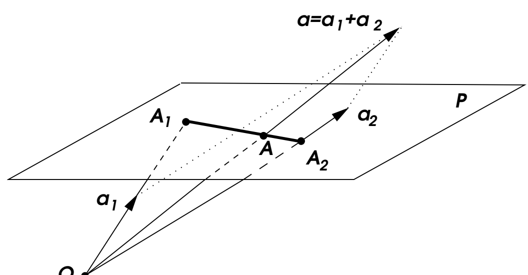

Figure 141

Position the plane P in space in such a way (Figure 141) that it misses the point O chosen for the origin. Then, to each point A on the plane, one can associate a line in space passing through the origin, namely the line OA. When the point comes equipped with a mass (or pseudo-mass) m, we associate to this material point on the plane the vector $\vec{a} = m\overrightarrow{OA}$ in space. We claim that this way, *the center of mass of a system of material points on the plane corresponds to the sum of the vectors associated to them in space.* Indeed, if A denotes the center of mass of a system of n material points $A_1, \ldots, A_n$ in the plane of masses $m_1, \ldots, m_n$, then the total mass is equal to

$m = m_1 + \cdots + m_n$, and the corresponding vector in space is

$$\vec{a} = m\overrightarrow{OM} = m_1\overrightarrow{OA_1} + \cdots + m_n\overrightarrow{OA_n} = \vec{a_1} + \cdots + \vec{a_n}.$$

In particular, the regrouping property of the center of mass follows from associativity of the addition of vectors.

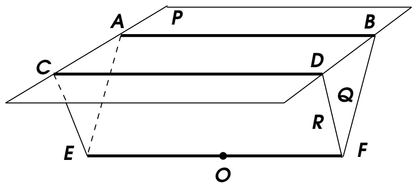

Figure 142

Remark. The above method of associating lines passing through the origin to points of the plane P turns out to be fruitful and leads to the so-called **projective geometry**. In projective geometry, beside ordinary points of the plane P, there exist "points at infinity." They correspond to lines passing through the origin and parallel to P (e.g. EF on Figure 142). Moreover, lines on the plane P (e.g. AB or CD) correspond to planes (Q or R) passing through the origin. When $AB\|CD$, the lines do not intersect on the plane P, but in projective geometry they intersect "at infinity," namely at the "point" corresponding to the line EF of intersection of the planes Q and R. Thus, the optical illusion that two parallel rails of a railroad track meet at the line of the horizon becomes reality in projective geometry.

EXERCISES

251. In the plane, let A, B, C, D, E be arbitrary points. Construct the point O such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE}$.

252.** In a circle, three non-intersecting chords AB, CD, and EF are given, each congruent to the radius of the circle, and the midpoints of the segments BC, DE, and FA are connected. Prove that the resulting triangle is equilateral.

253. Prove that if a polygon has several axes of symmetry, then they are concurrent.

254. Prove that the three segments connecting the midpoints of opposite edges of a tetrahedron bisect each other.

255. Prove that bisectors of exterior angles of a triangle meet extensions of the opposite sides at collinear points.