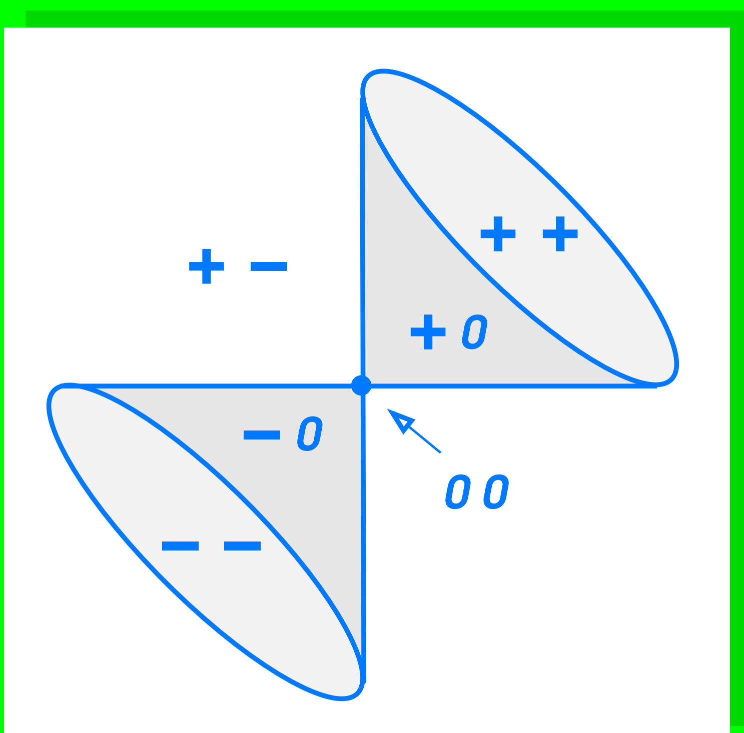
Linear Algebra



Alexander Givental

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by

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Preface

Here is a typical conversation of a student with professor which I frequently have in my capacity of a major adviser.

`S`: I have already taken the upper-division linear algebra.
`P`: Good! What what the course about?
`S`: Matrices?
`P`: Well, matrices are *things*. There is no point in introducing more things unless one can say something important about them. So, what were the *results* about matrices?
`S`: Diagonalization?
`P`: Right. Anything else?
`S`: There were many theorems, I don't remember all of them.
Then we start estimating how many. Say, a typical textbook might have 7 chapters, 5 sections each, with 3 theorems per section: some 100+ theorems in total. The problem I have with this count is this: modulo some variations on the same topics, I myself know only 4.

The four theorems give exhaustive answers to four problems of classification: of linear maps between two vector spaces (the Rank Theorem and some close relatives including the Gaussian elimination); of real (variations: complex, Hermitian) quadratic forms (the Inertia Theorem); of pairs of quadratic (or Hermitian) forms of which one is positive definite (the Orthogonal Diagonalization Theorem, or slightly more generally, the Spectral Theorem for normal operators); of linear maps from a (complex or real) vector space to itself (the Jordan Canonical Form Theorem).

This point of view on the traditional content of linear algebra as the classification theory of several basic geometrical objects up to suitable linear changes of coordinates greatly simplifies the logical structure of the subject. Application-wise it instills the right intuition by showing that there are only that many useful tools in the toolbox; without them anything else remains a useless triviality.

It is also in line with the 'higher', 'scientific' standpoint of modern mathematics. Namely, according to a paradigm developed in the 70-ies a (non-linear) problem is considered solved if reduced to a 'problem of linear algebra', i.e. a problem of classification of linear-algebraic data. (Examples of such reductions include some descriptions of instanton solutions to the Yang-Mills equations in mathematical physics, and classification of vector bundles over projective spaces in algebraic geometry). The theory of *quivers* outlined at the end of this book illustrates well what the modern-day problems of classification of linear-algebraic data entail.

Of course, mathematicians are fully aware of the nature of linear algebra as a classification theory, and implicitly this idea is present in the textbooks. However, it usually gets lost on its way to the mind of the reader through the woods of the 100+ theorems. In this book, we take the idea out of the woods.

While it is natural to develop the theory of vector spaces over an arbitrary field $\mathbb{K}$, for most applications in engineering and economics working over real and complex numbers would suffice. For the sake of versatility we attempt to do both. In the core material (typeset in this normal font) the reader may assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and be sure that we avoid some abstractions (such as the notion of quotient spaces). Alternatively, the reader may assume that $\mathbb{K}$ is any field, and brave the material typeset in a smaller dark green font. There the exposition may become more abstract.

The stand-alone section on *tensors*, which is not related to classification problems, was included merely for completeness's sake, and also out of desire to stress the role of determinants in multi-dimensional integration. Much of this section is also colored green, while the essay on differential forms assumes familiarity with vector calculus.

Among 450+ exercises many are provided with hints and/or answers, as indicated by the signs $\lightning$ and ✓ respectively. More difficult exercises are marked with *. We don't set up definitions of new terms (which are many) as separate paragraphs. When you see a word or phrase typeset in **boldface**, this is a new term, and the sentence is its definition. It is linked to the Index.

I should add that this course is self-contained in the sense that no prior knowledge of any aspect of linear algebra is expected.

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May 2022

Contents

Linear Algebra

Chapter 1

Introduction

One of our goals in this book is to equip the reader with a unifying view of linear algebra, or at least of what is traditionally studied under this name in university courses. With this mission in mind, we start with a *preview* of the subject, and describe its main achievements in lay terms.

To begin with a few words of praise: linear algebra is a very simple and useful subject, underlying most of other areas of mathematics, as well as its applications to physics, computer science, engineering, and economics. What makes linear algebra useful and efficient is that it provides ultimate solutions to several important mathematical problems. Furthermore, as should be expected of a truly fruitful mathematical theory, the problems it solves can be formulated in a rather elementary language, and make sense even before any advanced machinery is developed. Even better, the *answers* to these problems can also be described in elementary terms (in contrast with the *justification* of those answers, which better be postponed until adequate tools are developed). Finally, those several problems we are talking about are similar in their nature; namely, they all have the form of problems of *classification* of very basic mathematical objects.

Yet unready to discuss the general idea of classification in mathematics, we start off with a geometric introduction to vectors, and a summary of complex numbers. Then we work out a non-trivial model example: classification of quadratic curves on the plane. Then, with this example in mind, we will be able to describe the idea of classification in its abstract form, and finally present in elementary, down-to-earth terms the main problems of linear algebra, and the answers to these problems. At that point, the layout of further material will also become clear.

3 A Model Example: Quadratic Curves

Conic sections

On the coordinate plane, consider points (x, y), satisfying an equation of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$$

Generally speaking, such points form a curve. The set of all solutions to the equation is called a **quadratic curve**, provided that not all of the coefficients a, b, c vanish.

Being a quadratic curve is a geometric property. Indeed, if the coordinate system is changed (say, rotated, stretched, or translated), the same curve will be described by a different equation, but the left-hand-side of the equation will remain a polynomial of degree 2.

Our goal in this section is to describe all possible quadratic curves geometrically (i.e. disregarding their positions with respect to coordinate systems); or, in other words, to *classify* quadratic equations in two variables up to suitable changes of the variables.

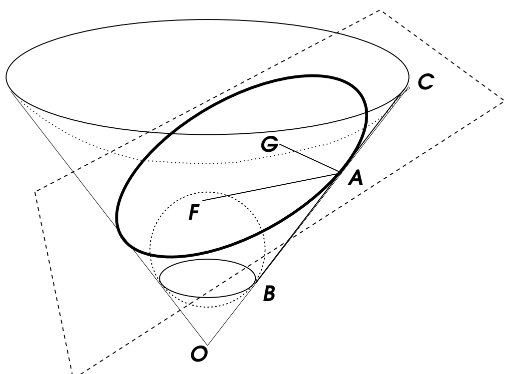

Figure 14

`Example:` **Dandelin's spheres**. The equation $x^2 + y^2 = z^2$ describes in a Cartesian coordinate system a cone (a half of which is shown on Figure 14). Intersecting the cone by planes, we obtain examples of quadratic curves. Indeed, substituting the equation $z = \alpha x + \beta y + \gamma$ of a section plane into the equation of the cone, we get a quadratic equation $x^2 + y^2 = (\alpha x + \beta y + \gamma)^2$ (which actually describes the projection of the conic section to the horizontal plane).

The conic section on the picture is an **ellipse**. According to one of many equivalent definitions,[6] an ellipse consists of all points of the plane with a fixed sum of the distances to two given points (called **foci** of the ellipse). Our picture illustrates an elegant way[7] to locate the foci of a conic section.

Place into the conic cup two balls (a small and a large one), and inflate the former and deflate the latter until they touch the plane (one from inside, the other from outside). Then the points F and G of the tangency are the foci.

Indeed, let A be an arbitrary point on the conic section. The segments AF and AG lie in the cutting plane and are therefore tangent to the balls at the points F and G respectively. On the generatrix OA, mark the points B and C where it crosses the circles of tangency of the cone with the balls. Then AB and AC are tangent at these points to the respective balls. All tangent segments from a given point to a given ball have the same length. Hence we find that $|AF| = |AB|$, and $|AG| = |AC|$. Therefore $|AF| + |AG| = |BC|$. But $|BC|$ is the distance along the generatrix between two parallel horizontal circles on the cone, and is the same for all generatrices. We conclude that the sum $|AF| + |AG|$ stays fixed when the point A moves along our conic section.

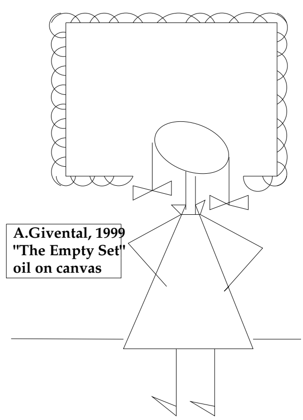

Figure 15. The Empty Set

Beside ellipses, we find among conic sections: **hyperbolas** (when a plane cuts through both halves of the cone), parabolas (cut by planes parallel to generatrices), and their degenerations (obtained

[6] According to a mock definition, "an ellipse is the circle inscribed into a square with unequal sides."

[7] Due to Germinal Pierre **Dandelin** (1794–1847).

when the cutting plane is replaced with the parallel one passing through the vertex O of the cone): just one point O, pairs of intersecting lines, and “double-lines.” We will see that this list exhausts all possible quadratic curves, except two degenerate cases: pairs of parallel lines and (yes!) empty curves.

EXERCISES

51. Prove that a hyperbolic conic section consists of all points on the section plane with a fixed *difference* of the distances to two points (called **foci**). Locate the foci by adjusting the construction of Dandelin’s spheres.

52.* Prove that light rays emitted from one focus of an ellipse and reflected in it as in a mirror will focus at the other focus. Formulate and prove similar optical properties of hyperbolas and parabolas. ϟ

53. Prove that the projections (Figure 16) of conic sections to the horizontal plane along the axis of the cone are quadratic curves.

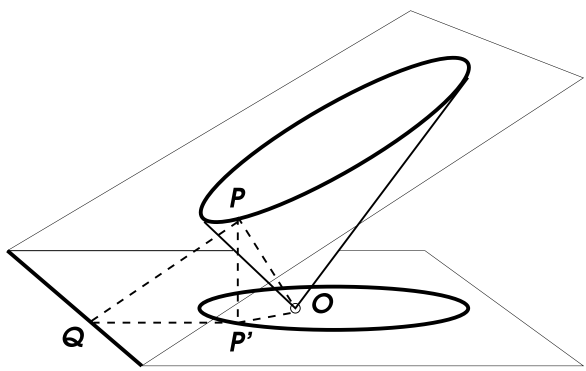

Figure 16

54.* Prove that the projections from the previous exercise can be characterized as plane curves formed by all points with a fixed ratio e (called **eccentricity**) between the distances to a fixed point (a **focus**) and a fixed line (called the **directrix**). ϟ

55.* Show that $e > 1$ for hyperbolas, $e = 1$ for parabolas, and $1 > e > 0$ for ellipses (e.g. $e = |P'O|/|P'Q|$ in Figure 16). ϟ

Orthogonal Diagonalization (toy version)

Let (x, y) be Cartesian coordinates on a Euclidean plane, and let Q be a **quadratic form** on the plane, i.e. a *homogeneous* degree-2 polynomial:

$$Q(x, y) = ax^2 + 2bxy + cy^2.$$

Theorem. ***Every quadratic form in a suitably rotated coordinate system assumes the form*** $Q = AX^2 + CY^2$.

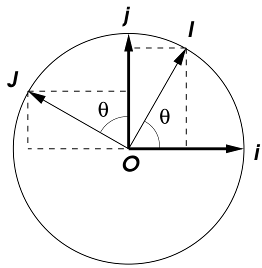

Figure 17

Proof. Rotating the unit coordinate vectors **i** and **j** counterclockwise through the angle θ (Figure 17), we obtain the following expressions for the unit coordinate vectors **I** and **J** of the rotated coordinate system:

$$\mathbf{I} = (\mathbf{cos}\,\theta)\mathbf{i} + (\mathbf{sin}\,\theta)\mathbf{j} \quad \text{and} \quad \mathbf{J} = -(\mathbf{sin}\,\theta)\mathbf{i} + (\mathbf{cos}\,\theta)\mathbf{j}.$$

Next, we express the radius-vector of any point in both coordinate systems:

$$x\mathbf{i} + y\mathbf{j} = X\mathbf{I} + Y\mathbf{J} = (X\,\mathbf{cos}\,\theta - Y\,\mathbf{sin}\,\theta)\mathbf{i} + (X\,\mathbf{sin}\,\theta + Y\,\mathbf{cos}\,\theta).$$

This shows that the old coordinates (x, y) are expressed in terms of the new coordinates (X, Y) by the formulas

$$x = X\,\mathbf{cos}\,\theta - Y\,\mathbf{sin}\,\theta, \quad y = X\,\mathbf{sin}\,\theta + Y\,\mathbf{cos}\,\theta. \qquad (*)$$

Substituting into $ax^2 + 2bxy + cy^2$, we rewrite the quadratic form in the new coordinates as $AX^2 + 2BXY + CY^2$, where A, B, C are certain expressions of a, b, c and θ. We want to show that choosing the rotation angle θ appropriately, we can make $2B = 0$. Indeed, making the substitution explicitly and ignoring X^2- and Y^2-terms, we find Q in the form

$$\cdots + XY\left(-2a\,\mathbf{sin}\,\theta\,\mathbf{cos}\,\theta + 2b(\mathbf{cos}^2\,\theta - \mathbf{sin}^2\,\theta) + 2c\,\mathbf{sin}\,\theta\,\mathbf{cos}\,\theta\right) + \dots$$

Thus $2B = (c - a)\,\mathbf{sin}\,2\theta + 2b\,\mathbf{cos}\,2\theta$. When $b = 0$, our task is trivial, as we can take $\theta = 0$. When $b \neq 0$, we can divide by $2b$ to obtain

$$\mathbf{cot}\,2\theta = \frac{a - c}{2b}.$$

Since **cot** assumes arbitrary real values, the theorem follows.

`Example.` For $Q = x^2+xy+y^2$, we have $\mathbf{cot}\, 2\theta = 0$, and find $2\theta = \pi/2 + \pi k$ ($k = 0, \pm 1, \pm 2, \dots$), i.e. up to multiples of 2π, $\theta = \pm\pi/4$ or $\pm 3\pi/4$. (This is a general rule: together with a solution θ, the angle $\theta + \pi$ as well as $\theta \pm \pi/2$, also work. Could you give an *a priori* explanation?) Taking $\theta = \pi/4$, we compute $x = (X - Y)/\sqrt{2}$, $y = (X + Y)/\sqrt{2}$, and finally find:

$$x^2 + y^2 + xy = X^2 + Y^2 + \frac{1}{2}(X^2 - Y^2) = \frac{3}{2}X^2 + \frac{1}{2}Y^2.$$

EXERCISES

56. A line is called an **axis of symmetry** of a given function $Q(x, y)$ if the function takes on the same values at every pair of points symmetric about this line. Prove that every quadratic form has two perpendicular axes of symmetry. (They are called **principal axes**.) ϟ

57. Prove that if a line passing through the origin is an axis of symmetry of a quadratic form $Q = ax^2 + 2bxy + cy^2$, then the perpendicular line is also its axis of symmetry.ϟ

58. Can a quadratic form on the plane have > 2 axes of symmetry? ✓

59. Find axes of symmetry of the following quadratic forms Q:

$(a)\ x^2 + xy + y^2,\quad (b)\ x^2 + 2xy + y^2,\quad (c)\ x^2 + 4xy + y^2.$

Which of them have level curves $Q = const$ ellipses? hyperbolas? ✓

60. Transform the equation $23x^2 + 72xy + 2y^2 = 25$ to one of the standard forms by rotating the coordinate system explicitly. ϟ ✓

Completing the squares

In our study of quadratic curves, the plan is to simplify the equation of the curve as much as possible by changing the coordinate system. In doing so we may assume that the coordinate system has already been rotated to make the coefficient at xy-term vanish. Therefore the equation at hands assumes the form

$$ax^2 + cy^2 + dx + ey + f = 0,$$

where a and c cannot both be zero. Our next step is based on **completing the squares**: whenever one of these coefficients (say, a) is non-zero, we can remove the corresponding linear term (dx) this way:

$$ax^2 + dx = a(x^2 + \frac{d}{a}x) = a\left((x + \frac{d}{2a})^2 - \frac{d^2}{4a^2}\right) = aX^2 - \frac{d^2}{4a}.$$

Here $X = x + d/2a$, and this change represents translation of the origin of the coordinate system from the point $(x, y) = (0, 0)$ to $(x, y) = (-d/2a, 0)$.

Example. The equation $x^2 + y^2 = 2ry$ can be rewritten by completing the square in y as $x^2 + (y - r)^2 = r^2$. Therefore, it describes the circle of radius r centered at the point $(0, r)$ on the y-axis.

With the operations of completing the squares in one or both variables, renaming the variables if necessary, and dividing the whole equation by a non-zero number (which does not change the quadratic curve), we are well-armed to obtain the classification.

Classification of Quadratic Curves

Case I: $a \neq 0 \neq c$. The equation is reduced to $aX^2 + cY^2 = F$ by completing the squares in each of the variables.

Sub-case (i): $F \neq 0$. Dividing the whole equation by F, we obtain the equation $(a/F)X^2 + (c/F)Y^2 = 1$. When both a/F and c/F are positive, the equation can be re-written as

$$\frac{X^2}{\alpha^2} + \frac{Y^2}{\beta^2} = 1.$$

This is the equation of an ellipse with **semiaxes** α and β (Figure 18). When one a/F and c/F have opposite signs, we get (possibly renaming the variables) the equation of a hyperbola (Figure 19)

$$\frac{X^2}{\alpha^2} - \frac{Y^2}{\beta^2} = 1.$$

When a/F and c/F are both negative, the equation has no real solutions, so that the quadratic curve is *empty* (Figure 15).

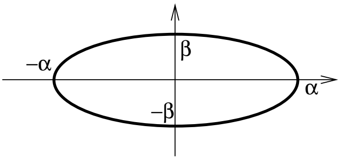

Figure 18. Ellipse

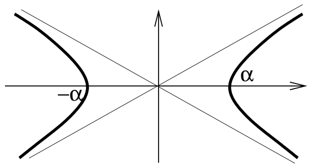

Figure 19. Hyperbola

Sub-case (ii): $F = 0$. Then, when a and c have opposite signs (say, $a = \alpha^2 > 0$, and $c = -\gamma^2 < 0$), the equation $\alpha^2 X^2 = \gamma^2 Y^2$ describes a pair of intersecting lines $Y = \pm kX$, where $k = \alpha/\gamma$ (Figure 20). When a and c are of the same sign, the equation $aX^2 + cY^2 = 0$ has only one real solution: $(X, Y) = (0, 0)$. The quadratic curve is a "thick" point.[8]

Case II: One of a, c ***is*** 0. We may assume without loss of generality that $c = 0$. Since $a \neq 0$, we can still complete the square in x to obtain an equation of the form $aX^2 + ey + F = 0$.

Sub-case (i): $e \neq 0$. Divide the whole equation by e and put $Y = y - F/e$ to arrive at the equation $Y = -aX^2/e$. This curve is a **parabola** $Y = kX^2$, where $k = -a/e \neq 0$ (Figure 21).

Sub-case (ii): $e = 0$. The equation $X^2 = -F/a$ describes: a pair of parallel lines $X = \pm k$ (where $k = \sqrt{-F/a}$), or the empty set (when $F/a > 0$), or a "double-line" $X = 0$ (when $F = 0$).

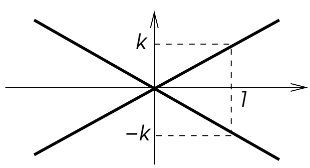

Figure 20

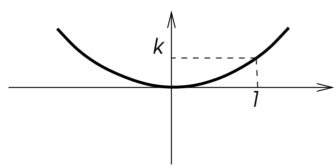

Figure 21

We have proved the following:

Theorem. ***Every quadratic curve on a Euclidean plane is one of the following: an ellipse, hyperbola, parabola, a pair of intersecting, parallel, or coinciding lines, a "thick" point or the empty set. In a suitable Cartesian coordinate system, the curve is described by one of the standard equations:***

$$\frac{X^2}{\alpha^2} \pm \frac{Y^2}{\beta^2} = 1, -1, \textit{ or } 0; \quad Y = kX^2; \quad X^2 = k.$$

[8] In fact this is the point of intersection of a pair of "imaginary" lines consisting of non-real solutions.

EXERCISES

61. Find the places of the following quadratic curves in our classification: $y = x^2 + x,\ xy = 1,\ xy = 0,\ xy = y,\ x^2 + x = y^2 - y,\ x^2 + x + y^2 - y = 0.$

62. Following the steps of our classification, reduce the quadratic equation $x^2 + xy + y^2 + \sqrt{2}(x - y) = 0$ to one of the standard forms. Show that the curve is an ellipse, and find its semiaxes. $\lightning$ $\checkmark$

63. Use our classification theorem to prove that, with the exception of parabolas, each conic section has a center of symmetry. $\lightning$

64. Locate foci of (a) ellipses and (b) hyperbolas given by the standard equations $x^2/\alpha^2 \pm y^2/\beta^2 = 1$, where $\alpha > \beta > 0$. $\checkmark$

65. Show that "renaming coordinates" can be accomplished by a linear geometric transformation on the plane. $\lightning$

66. Prove that ellipses are obtained by stretching (or shrinking) unit circles in two perpendicular directions with two different coefficients.

67. From the Orthogonal Diagonalization Theorem on the plane, derive the following **Inertia Theorem** for quadratic forms in two variables: Every quadratic form on the plane in a suitable (not necessarily Cartesian) coordinate system assumes one of the forms:

$$X^2 + Y^2,\ X^2 - Y^2,\ -X^2 - Y^2,\ X^2,\ -Y^2,\ 0.$$

Sketch graphs of these functions.

68. Complete the squares to find out which of the following curves are ellipses and which are hyperbolas: $\lightning$ $\checkmark$
$x^2 + 4xy = 1,\ x^2 + 2xy + 4y^2 = 1,\ x^2 + 4xy + 4y^2 = 1,\ x^2 + 6xy + 4y^2 = 1.$

69. Show that a quadratic form $ax^2 + 2bxy + cy^2$ is, up to a sign $\pm$, the square $(\alpha x + \beta y)^2$ of a linear function if and only if $ac = b^2$. $\lightning$

70. Show that if, in addition to rotation, reflection, translation of coordinate systems, and multiplication of a quadratic equation by a non-zero constant, the change of scales of the coordinates is also allowed, then each quadratic equation can be transformed to one of the following 9 normal forms:

$$x^2 + y^2 = 1,\ x^2 + y^2 = 0,\ x^2 + y^2 = -1,\ x^2 - y^2 = 1,\ x^2 - y^2 = 0,$$
$$x^2 = y,\quad x^2 = 1,\quad x^2 = 0,\quad x^2 = -1.$$

71. Examine the curves defined by the above equations to conclude that they fall into 8 different types.

72. Find the place of $x^2 - 4y^2 = 2x - 4y$ in the classification of quadratic curves. $\checkmark$

4 Problems of Linear Algebra

Classifications in mathematics

Classifications are intended to bring order into seemingly complex or chaotic matters. Yet, there is a major difference between, say, our classification of quadratic curves and Carl **Linnaeus**' *Systema Naturae*.

For two quadratic curves to be in the same *class*, it is not enough that they share a number of features. What is required is a *transformation* of a prescribed type that would transform one of the curves into the other, and thus make them **equivalent** in this sense, i.e. the same *up to* such transformations.

What types of transformations are allowed (e.g., changes to *arbitrary* new coordinate systems, or only to *Cartesian* ones) may be a matter of choice. With every choice, the classification of objects of a certain kind (i.e. quadratic curves in our example) *up to* transformations of the selected type becomes a well-posed mathematical problem.

A complete answer to a classification problem should consist of
– a list of **normal** (or **canonical**) **forms**, i.e. representatives of the classes of equivalence, and
– a **classification theorem** establishing that each object of the kind (quadratic curve in our example) is equivalent to exactly one of the normal forms, i.e. in other words, that
  (i) each object can be transformed into a normal form, and
  (ii) no two normal forms can be transformed into each other.

Simply put, Linear Algebra deals with classifications of linear and/or quadratic equations, or systems of such equations. One might think that all that equations do is ask: *Solve us!* Unfortunately this attitude toward equations does not lead too far. It turns out that very few equations (and kinds of equations) can be explicitly *solved*, but all can be *studied* and many *classified*.

The idea is to replace a given "hard" (possibly unsolvable) equation with another one, the normal form, which should be chosen to be as "easy" as it is possible to find in the same equivalence class. Then the normal form should be studied (and hopefully "solved") thus providing information about the original "hard" equation.

What sort of information? Well, *any* sort that remains *invariant* under the equivalence transformations in question.

For example, in classification of quadratic curves up to changes of Cartesian coordinate systems, all equivalent ellipses are indistinguishable from each other *geometrically* (in particular, they have the same semiaxes) and differ only by the choice of a Cartesian coordinate system. However, if arbitrary rescaling of coordinates is also allowed, then all ellipses become indistinguishable from circles (but still different from hyperbolas, parabolas, etc.)

Whether a classification theorem really simplifies the matters, depends on the kind of objects in question, the chosen type of equivalence transformations, and the applications in mind. In practice, the problem often reduces to finding sufficiently simple normal forms and studying them in great detail.

The subject of linear algebra fits well into the general philosophy just outlined. Below, we formulate four model classification problems of linear algebra, solve them by bare hands in the simplest case of dimension 1, and state the respective general answers. Together with a number of variations and applications, which will be presented later in due course, these problems form what is usually considered the main course of linear algebra.

The Rank Theorem

`Question.` *Given m linear functions in n variables,*

$$\begin{array}{rcl} y_1 & = & a_{11}x_1 + ... + a_{1n}x_n \\ & ... & \\ y_m & = & a_{m1}x_1 + ... + a_{mn}x_n \end{array},$$

what is the simplest form to which they can be transformed by linear changes of the variables,

$$\begin{array}{rcl} y_1 & = & b_{11}Y_1 + ... + b_{1m}Y_m \\ & ... & \\ y_m & = & b_{m1}Y_1 + ... + b_{mm}Y_m \end{array}, \quad \begin{array}{rcl} x_1 & = & c_{11}X_1 + ... + c_{1n}X_n \\ & ... & \\ x_n & = & c_{n1}X_1 + ... + c_{nn}X_n \end{array} \quad ?$$

`Example.` Consider a linear function in one variable: $y = ax$. We are allowed to make substitutions $y = bY$ and $x = cX$, where however $b \neq 0$ and $c \neq 0$ (so that we could reverse the substitutions). The substitutions will result in a new, transformed function: $Y = b^{-1}acX$. Clearly, if $a = 0$, then no matter what substitution we make, the linear function will remain identically zero. On the other hand, if $a \neq 0$, we can choose such values of b and c that the coefficient $b^{-1}ac$

becomes equal to 1 (e.g. take $b = 1$ and $c = a^{-1}$). Thus, every linear function $y = ax$ is either identically zero: $Y = 0$, or can be transformed to $Y = X$.

`Theorem.` ***Every system of m linear functions in n variables can be transformed by suitable linear changes of dependent and independent variables to exactly one of the normal forms:***

$$Y_1 = X_1, \quad \ldots, \quad Y_r = X_r, \quad Y_{r+1} = 0, \quad \ldots, \quad Y_m = 0,$$

where $0 \leq r \leq m, n$.

The number r featuring in the answer is called the **rank** of the given system of m linear functions.

EXERCISES

73. Transform explicitly one linear function $y = -3x$ to the normal form prescribed by the Rank Theorem.

74. The same for the linear function $y = 3x_1 - 2x_2$.

75. The same for the system: $y_1 = x_1 + x_2$, $y_2 = x_1 - x_2$.

76. Use the Rank Theorem to prove that if two systems of m linear functions in n variables have the same rank then they can be transformed into each other by linear changes of dependent and independent variables. ↯

The Inertia Theorem

`Question.` *Given a* **quadratic form** *(i.e. a homogeneous quadratic function) in n variables,*

$$Q = q_{11}x_1^2 + 2q_{12}x_1x_2 + 2q_{13}x_1x_3 + ... + q_{nn}x_n^2,$$

what is the simplest form to which it can be transformed by a linear change of the variables

$$\begin{array}{rcl} x_1 & = & c_{11}X_1 + ... + c_{1n}X_n \\ & ... & \\ x_n & = & c_{n1}X_1 + ... + c_{nn}X_n \end{array} \quad ?$$

`Example.` A quadratic form in one variable, x, has the form qx^2. A substitution $x = cX$ (with $c \neq 0$), transforms it into qc^2X^2. Of course, if $q = 0$, no substitution will change the fact that the function is identically zero. When $q \neq 0$, we can make the absolute value of coefficient qc^2 equal to 1 (by choosing $c = \pm\sqrt{|q^{-1}|}$). However, no substitution will change the sign of the coefficient (that is, a positive quadratic form will remain positive, and negative will remain negative). Thus, every quadratic form in one variable can be transformed to exactly one of these: X^2, $-X^2$, or 0.

`Theorem.` ***Every quadratic form in n variables can be transformed by a suitable linear change of the variables to exactly one of the normal forms:***

$$X_1^2 + ... + X_p^2 - X_{p+1}^2 - ... - X_{p+q}^2 \quad \textit{where } 0 \le p + q \le n.$$

Note that, in a way, the theorem claims that the n-dimensional case can be reduced to the sum (we will later call it "**direct sum**") of n one-dimensional answers found in the example: X^2, $-X^2$, or 0. The possibility of such reduction of a higher-dimensional problem to the direct sum of one-dimensional problems is a standard theme of linear algebra.

The numbers p and q of positive and negative squares in the normal form are called **inertia indices** of the quadratic form in question. If the quadratic form Q is known to be positive everywhere outside the origin, the Inertia Theorem tells us that in a suitable coordinate system Q assumes the form $X_1^2 + ... + X_n^2$, i.e. its inertia indices are $p = n,\ q = 0$.

EXERCISES

77. Transform explicitly the quadratic forms $4x^2$ and $-9y^2$ to their normal forms prescribed by the Inertia Theorem.

78. Transform the quadratic forms from the previous exercise into each other by a substitution $x = cy$ with possibly complex value of c.

79. Classify quadratic forms $Q = ax^2$ in one variable with *complex* coefficients (i.e. $a \in \mathbb{C}$) up to complex linear changes: $x = cX,\ c \in \mathbb{C}, c \neq 0$. ✓

80.* In the Inertia Theorem with $n = 2$, show that that there are 6 normal forms, and prove that they are pairwise non-equivalent. ϟ

81. Find the indices of inertia of the quadratic form $Q(x, y) = xy$. ϟ

82. Show that $X_1^2 + ... + X_n^2$ is the only one of the normal forms of the Inertia Theorem which is positive everywhere outside the origin.

83. Sketch the level surfaces $Q(X_1, X_2, X_3) = \textit{const}$ for all normal forms in the Inertia Theorem with $n = 3$.

84. How many normal forms are there in the Inertia Theorem for quadratic forms in n variables? ✓

The Orthogonal Diagonalization Theorem

`Question.` *Given two homogeneous quadratic forms in* n *variables,* $Q(x_1, ..., x_n)$ *and* $S(x_1, ..., x_n)$*, of which the first one is known to be positive everywhere outside the origin, what is the simplest form to which they can be simultaneously transformed by a linear change of the variables?*

`Example.` In the case $n = 1$, we have $Q(x) = qx^2$, where $q > 0$, and $S(x) = sx^2$, where s is arbitrary. As we know, the first quadratic form is transformed by the substitution $x = q^{-1/2}X$ into X^2. The same transformation will change S into λX^2 with $\lambda = sq^{-1}$. Of course, one can make S to be $\pm\tilde{X}^2$ (if $s \neq 0$) by rescaling the variable once again, but this may destroy the form X^2 of the function Q. In fact the only substitutions $X = C\tilde{X}$ which *preserve* Q (i.e. don't change the coefficient) are those with $C = \pm 1$. Unfortunately such substitutions do not affect at all the coefficient λ in the function S: $\lambda X^2 = \lambda(\pm\tilde{X})^2 = \lambda\tilde{X}^2$. We conclude that each pair Q, S can be transformed into one of the pairs $X^2, \lambda X^2$, where λ is a real number, but two such pairs with different values of λ cannot be transformed into each other.

`Theorem.` ***Every pair*** Q, S ***of quadratic forms in*** n ***variables, of which*** Q ***is positive everywhere outside the origin, can be transformed by a linear change of the variables into exactly one of the normal forms***

$$Q = X_1^2 + ... + X_n^2,\ \ S = \lambda_1 X_1^2 + ... + \lambda_n X_n^2,\ \ \textbf{\textit{where}}\ \lambda_1 \geq ... \geq \lambda_n.$$

The real numbers $\lambda_1, \dots, \lambda_n$ are called **eigenvalues** of the given pair of quadratic forms (and are often said to form their **spectrum**).

Note that this theorem, too, reduces the n-dimensional problem to the "direct sum" of n one-dimensional problems solved in our Example.

EXERCISES

85. Prove the Orthogonal Diagonalization Theorem for $n = 2$ using results of Section 3. ϟ

86. Transform explicitly the quadratic form $Q = 3x^2 + 16y^2 + 9z^2$ to its normal form prescribed by the Inertia theorem, and apply the same transformation to the quadratic form $S = x^2 - 4y^2 + 12yz$.

87. Find the spectrum of the pair of quadratic forms: $Q = 3x^2+16y^2+9z^2$, $S = x^2 - 4y^2 + 12z^2$.

The Jordan Canonical Form Theorem

The fourth question deals with a system of n linear functions in n variables. Such an object is the special case of systems of m functions in n variables when $m = n$. According to the Rank Theorem, such a system of rank $r \leq n$ can be transformed to the form $Y_1 = X_1, \ldots, Y_r = X_r, Y_{r+1} = \cdots = Y_n = 0$ by linear changes of dependent and independent variables. There are many cases however where relevant information about the system is lost when dependent and independent variables are changed *separately*. This happens whenever both groups of variables describe objects in the same space (rather than in two different ones).

An important class of examples comes from the theory of Ordinary Differential Equations (ODE for short).

`Example.` Consider a linear first order ODE $\dot{x} = \lambda x$. It relates the values $x(t)$ of an unknown function, x, with its rate of change in time, $\dot{x}$ (which is the short notation for dx/dt). A rescaling of the function by $x = cX$ would make little sense if not accompanied with the simultaneous rescaling of the rate, $\dot{x} = c\dot{X}$ (we assume that the rescaling coefficient c is time-independent). Unfortunately, such a rescaling does not affect the form of the equation: $\dot{X} = c^{-1}\lambda cX = \lambda X$. We conclude that no two linear first order ODEs $\dot{x} = \lambda x$ with different values of the coefficient λ can be transformed into each other by a linear change of the variable.

We will describe the fourth classification problem in the context of the ODE theory, although it can be stated more abstractly as a problem about n linear functions in n variables, to be transformed by a single linear change acting on both dependent and independent variables *the same way*.

`Question.` *Given a system of n linear homogeneous 1st order constant coefficient ODEs in n unknowns:*

$$\begin{array}{rcl} \dot{x}_1 & = & a_{11}x_1 + \ldots + a_{1n}x_n \\ & \ldots & \\ \dot{x}_n & = & a_{n1}x_1 + \ldots + a_{nn}x_n \end{array},$$

what is the simplest form to which it can be transformed by a linear change of the unknowns:

$$\begin{array}{rcl} x_1 & = & c_{11}X_1 + \ldots + c_{1n}X_n \\ & \ldots & \\ x_n & = & c_{n1}X_1 + \ldots + c_{nn}X_n \end{array} \quad ?$$

There is an advantage in answering this question *over* $\mathbb{C}$, i.e. assuming that the coefficients c_{ij} in the change of variables, as well as the coefficients a_{ij} of the given ODE system are allowed to be complex numbers. The advantage is due to the unifying power of the Fundamental Theorem of Algebra, discussed in Supplement "Complex Numbers."

`Example.` Consider a single mth order linear ODE of the form:

$$(\frac{d}{dt} - \lambda)^m y = 0, \quad \text{where } \lambda \in \mathbb{C}.$$

By setting

$$y = x_1, \ \frac{d}{dt}y - \lambda y = x_2, \ (\frac{d}{dt} - \lambda)^2 y = x_3, \ \ldots, \ (\frac{d}{dt} - \lambda)^{m-1} y = x_m,$$

the equation can be written as the following system of m ODEs of the 1st order:

$$\begin{array}{lcllllll} \dot{x}_1 & = & \lambda x_1 & + & x_2 & & & \\ \dot{x}_2 & = & & & \lambda x_2 & + & x_3 & \\ & & & \cdots & & & & \\ \dot{x}_{m-1} & = & & & & & \lambda x_{m-1} & + \ x_m \\ \dot{x}_m & = & & & & & & \lambda x_m \end{array} .$$

Let us call this system the **Jordan block** of size m with the eigenvalue λ. Introduce a **Jordan system** of several Jordan blocks of sizes $m_1, ..., m_r$ with the eigenvalues $\lambda_1, ..., \lambda_r$. It can be similarly compressed into the system

$$(\frac{d}{dt} - \lambda_1)^{m_1} y_1 = 0, \ ..., \ (\frac{d}{dt} - \lambda_r)^{m_r} y_r = 0$$

of r *unlinked* ODEs of the orders $m_1, ..., m_r$.

The numbers $\lambda_1, \ldots, \lambda_r$ here are not assumed to be necessarily distinct. In fact they are the roots of a certain degree n polynomial, $p(\lambda) = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_r)^{m_r}$, (called the **characteristic polynomial**), which can be associated with every linear ODE system, and does not change under the linear changes of the unknowns. When the polynomial has all its n roots distinct (that is, all $m_i = 1$, and $r = n$), the Jordan system assumes the form $\dot{x}_1 = \lambda_1 x_1, \ \ldots, \ \dot{x}_n = \lambda_n x_n$ of n unlinked first order ODEs discussed in our one-dimensional example. However, the theorem below implies that *not every* linear ODE system can be reduced to such a superposition (or direct sum) of one-dimensional ODEs. In particular, a single Jordan block of size $m > 1$ cannot be transformed into the superposition of one-dimensional ODEs.

`Theorem.` ***Every constant coefficient system of*** n ***linear*** 1***st order ODEs in*** n ***unknowns can be transformed by a complex linear change of the unknowns into exactly one (up to reordering of the blocks) of the Jordan systems with*** $m_1 + \ldots + m_r = n$.

EXERCISES

88. Find the general solution to the differential equation $\dot{x} = \lambda x$. ✓

89. Find the general solution to the system of ODE: $\dot{x} = 3x$, $\dot{y} = -y$, $\dot{z} = 0$. ✓

90. Verify that $y(t) = e^{\lambda t}\left(c_0 + tc_1 + \ldots + c_{m-1}t^{m-1}\right)$, where $c_i \in \mathbb{C}$ are arbitrary constants, is the general solution to the ODE $(\frac{d}{dt} - \lambda)^m y = 0$.

91. Rewrite the **pendulum equation** $\ddot{x} = -x$ as a system. ϟ

92.* Identify the Jordan form of the system $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1$. ✓

93.* Find the general solution to the system $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$, and sketch the trajectories $(x_t(t), x_2(t))$ on the plane. Prove that the system cannot be transformed into any system $\dot{y}_1 = \lambda_1 y_1$, $\dot{y}_2 = \lambda_2 y_2$ of two unlinked ODEs.

Fools and wizards

In the rest of the book we will undertake a more systematic study of the four basic problems and prove the classification theorems stated here. However, the reader (not unlike a fairy-tale hero) should be prepared to meet the following three challenges of the next Chapter.

Firstly, linear algebra has developed an adequate language, based on the abstract notion of **vector space**. It allows one to represent relevant mathematical objects and results in ways much less cumbersome and thus more efficient than those found in the previous discussion. This language is introduced at the beginning of Chapter 2. The challenge here is to get accustomed to the abstract way of thinking.

Secondly, one will find there much more diverse material than what has been described in the Introduction. This is because many mathematical objects and classification problems about them can be *reduced* (speaking loosely or literally) to the four problems discussed above. The challenge is to learn how to recognize situations where results of linear algebra can be helpful. Many of those objects will be introduced in the second section of Chapter 2.

Finally, we will encounter one more fundamental result of linear algebra, which is not a classification, but an important (and beautiful) formula. It answers the question: *Which substitutions of the form*

$$\begin{array}{rcl} x_1 & = & c_{11}X_1 + ... + c_{1n}X_n \\ & ... & \\ x_n & = & c_{n1}X_1 + ... + c_{nn}X_n \end{array}$$

are indeed changes of the variables and can therefore be inverted by expressing $X_1, ..., X_n$ *linearly in terms of* $x_1, ..., x_n$*, and how to describe such inversion explicitly?* The answer is given in terms of the **determinant**, a remarkable function of n^2 variables $c_{11}, ..., c_{nn}$, which will also be studied in Chapter 2.

Let us describe now the principle by which our four main themes are grouped in Chapters 3 and 4.

Note that Jordan canonical forms and the normal forms in the Orthogonal Diagonalization Theorem do not form discrete lists, but instead depend on continuous parameters — the eigenvalues. Based on experience with many mathematical classifications, it is considered that the number of parameters on which equivalence classes in a given problem depend, is the right measure of complexity of the classification problem. Thus, Chapter 3 deals with **simple problems** of Linear Algebra, i.e. those classification problems where equivalence classes do not depend on continuous parameters. Respectively, the non-simple problems are studied in Chapter 4.

Finally, let us mention that the proverb: *Fools ask questions that wizards cannot answer,* fully applies in Linear Algebra. In addition to the four basic problems, there are many similarly looking questions that one can ask: for instance, to classify *triples* of quadratic forms in n variables up to linear changes of the variables. In fact, in this problem, the number of parameters, on which equivalence classes depend, grows with n at about the same rate as the number of parameters on which the three given quadratic forms depend. We will have a chance to touch upon such problems of Linear Algebra in the last, Epilogue section, in connection with *quivers.* The modern attitude toward such problems is that they are *unsolvable.*

EXERCISES

94. Using results of Section 3, derive the Inertia Theorem for $n = 2$.

95. Show that classification of real quadratic curves up to linear inhomogeneous changes of coordinates consists of 8 equivalence classes, but if the coordinate systems are required to remain Cartesian, then there are infinitely many equivalence classes, which depend on 2 continuous parameters.

96. Is there any difference between classification of quadratic equations $F(x, y) = 0$ up to linear inhomogeneous coordinate changes and multiplication of the equations by non-zero constants, and of quadratic curves $\{(x, y) \mid F(x, y) = 0\}$ up to the same type of coordinate transformations? ✓

97.* From the Orthogonal Diagonalization Theorem (as it is stated in this Section) in the case $n = 2$, derive the "toy version" proved in Section 3. ϟ

98. Let us represent a quadratic form $ax^2 + 2bxy + cy^2$ by the point (a, b, c) in the 3-space. Show that the surface $ac = b^2$ is a cone. ϟ

99. Locate the 6 normal forms $(x^2+y^2, x^2-y^2, -x^2-y^2, x^2, -y^2, 0)$ of the Inertia Theorem with respect to the cone $ac = b^2$ on Figure 22.

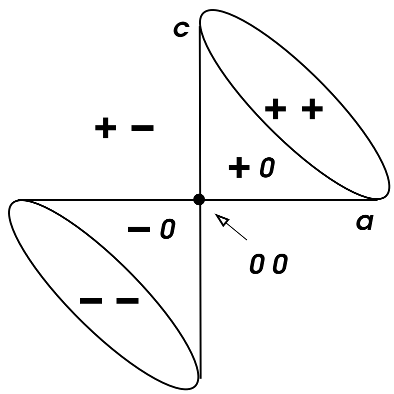

Figure 22

100. The cone $ac = b^2$ divides the 3-space into three regions (Figure 22). Show that these three regions, together with the two branches of the cone itself, and the origin form the partition of the space into 6 parts which exactly correspond to the 6 equivalence classes of the Inertia Theorem in dimension 2.

101. How many arbitrary coefficients are there in a quadratic form in n variables? ✓

102.* Show that equivalence classes of *triples* of quadratic forms in n variables must depend on at least $n^2/2$ parameters. ϟ

3 Determinants

Definition

Let A be a *square* matrix of size n:

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ & \ldots & \\ a_{n1} & \ldots & a_{nn} \end{bmatrix}$$

Its **determinant** is a *scalar* $\det A$ defined by the formula

$$\det A = \sum_{\sigma} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \ldots a_{n\sigma(n)}.$$

Here σ is a **permutation** of the indices $1, 2, \ldots, n$. A permutation σ can be considered as an invertible function $i \mapsto \sigma(i)$ from the set of n elements $\{1, \ldots, n\}$ to itself. We use the functional notation $\sigma(i)$ in order to specify the i-th term in the permutation $\sigma = \begin{pmatrix} 1 & \ldots & n \\ \sigma(1) & \ldots & \sigma(n) \end{pmatrix}$. Thus, each **elementary product** in the determinant formula contains exactly one matrix entry from each row, and these entries are chosen from n different columns. The sum is taken over all $n!$ ways of making such choices. The coefficient $\varepsilon(\sigma)$ in front of the elementary product equals 1 or -1 and is called the **sign** of the permutation σ.

We will explain the general rule of the signs after a few examples. In these examples, we begin using one more conventional notation for determinants. According to it, a square array of matrix entries placed between two vertical bars denotes the *determinant* of the matrix. Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denotes a *matrix*, but $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes a *number* equal to the determinant of that matrix.

`Examples.` (1) For $n = 1$, the determinant $|a_{11}| = a_{11}$.

(2) For $n = 2$, we have: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

(3) For $n = 3$, we have $3! = 6$ summands

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32}$ corresponding to permutations $\binom{123}{123}$, $\binom{123}{213}$, $\binom{123}{231}$, $\binom{123}{321}$, $\binom{123}{312}$, $\binom{123}{132}$.

The rule of signs for $n = 3$ is schematically shown on Figure 28.

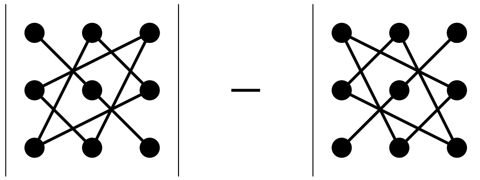

Figure 28

EXERCISES

203. Prove that the following determinant is equal to 0:

$$\begin{vmatrix} 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \\ 0 & 0 & 0 & e & f \\ p & q & r & s & t \\ v & w & x & y & z \end{vmatrix}. \qquad ↯$$

204. Compute determinants:

$$\begin{vmatrix} \mathbf{cos}\, x & -\mathbf{sin}\, x \\ \mathbf{sin}\, x & \mathbf{cos}\, x \end{vmatrix}, \quad \begin{vmatrix} \mathbf{cosh}\, x & \mathbf{sinh}\, x \\ \mathbf{sinh}\, x & \mathbf{cosh}\, x \end{vmatrix}, \quad \begin{vmatrix} \mathbf{cos}\, x & \mathbf{sin}\, y \\ \mathbf{sin}\, x & \mathbf{cos}\, y \end{vmatrix}. \qquad \checkmark$$

205. Compute determinants:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{vmatrix}. \qquad \checkmark$$

The parity of permutations

The general rule of signs relies on properties of permutations.

Let Δ_n denote the following polynomial in n variables $x_1, \dots, x_n$:

$$\Delta_n(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j).$$

`Examples:` $\Delta_2 = x_1 - x_2$, $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. By definition, $\Delta_1 = 1$. In general, Δ_n is the product of all "n-choose-2" linear factors $x_i - x_j$ written in such a way that $i < j$.

Let σ be a permutation of $\{1, \dots, n\}$. It acts on polynomials P in the variables $x_1, \dots, x_n$ by permutation of the variables: $(\sigma P)(x_1, \dots, x_n) := P(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

`Example.` Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Then

$$\sigma\Delta_3 = (x_3-x_1)(x_3-x_2)(x_1-x_2) = (-1)^2(x_1-x_3)(x_2-x_3)(x_1-x_2).$$

One says that σ **inverses** a pair of indices $i < j$ if $\sigma(i) > \sigma(j)$. The total number $l(\sigma)$ of pairs $i < j$ that σ inverses is called the **length** of the permutation σ. Thus, in the previous example, σ inverses the pairs $(1, 2)$ and $(1, 3)$, and has length $l(\sigma) = 2$.

`Lemma.` $\sigma\Delta_n = \varepsilon(\sigma)\Delta_n$, ***where*** $\varepsilon(\sigma) = (-1)^{l(\sigma)}$.

`Proof.` Indeed, a permutation of $\{1, \dots, n\}$ also permutes all pairs $i \neq j$, and hence permutes all the linear factors in Δ_n. However, a factor $x_i - x_j$ is transformed into $x_{\sigma(i)} - x_{\sigma(j)}$, which occurs in the product Δ_n with the same sign whenever $\sigma(i) < \sigma(j)$, and with the opposite sign whenever $\sigma(i) > \sigma(j)$. Thus, $\sigma\Delta_n$ differs from Δ_n by the sign $(-1)^{l(\sigma)}$. □

A permutation σ is called **even** or **odd** depending on the sign $\varepsilon(\sigma)$, i.e. when the length is even or odd respectively.

`Examples.` (1) The **identity permutation** id (defined by $\text{id}(i) = i$ for all i) is even since $l(\text{id}) = 0$.

(2) Consider a **transposition** τ, i.e. a permutation that swaps two indices, say $i < j$, leaving all other indices in their respective places. Then $\tau(j) < \tau(i)$, i.e. τ inverses the pair of indices $i < j$. Besides, for every index k such that $i < k < j$ we have: $\tau(j) < \tau(k) < \tau(i)$, i.e. both pairs $i < k$ and $k < j$ are inverted. Note that all other pairs of indices are not inverted by τ, and hence $l(\tau) = 2(j - i) + 1$. In particular, *every transposition is odd*: $\varepsilon(\tau) = -1$.

`Proposition.` ***Composition of two even or two odd permutations is even, and composition of one even and one odd permutation is odd:*** $\varepsilon(\sigma\sigma') = \varepsilon(\sigma)\varepsilon(\sigma')$.

`Proof.` We have:

$$\varepsilon(\sigma\sigma')\Delta_n := (\sigma\sigma')\Delta_n = \sigma(\sigma'\Delta_n) = \varepsilon(\sigma')\sigma\Delta_n = \varepsilon(\sigma')\varepsilon(\sigma)\Delta_n.$$

`Corollary 1.` ***Inverse permutations have the same parity.***

`Corollary 2.` ***Whenever a permutation is written as the product of transpositions, the parity of the number of the transpositions in the product remains the same and coincides with the parity of the permutation: If*** $\sigma = \tau_1 \dots \tau_N$, ***then*** $\varepsilon(\sigma) = (-1)^N$.

Here are some illustrations of the above properties in connection with the definition of determinants.

`Examples.` (3) The transposition (21) is odd. That is why the term $a_{12}a_{21}$ occurs in 2×2-determinants with the negative sign.

(4) The permutations $\binom{123}{123}$, $\binom{123}{213}$, $\binom{123}{231}$, $\binom{123}{321}$, $\binom{123}{312}$, $\binom{123}{132}$ have lengths $l=0,1,2,3,2,1$ and respectively signs $\varepsilon=1,-1,1,-1,1,-1$ (thus explaining Figure 28). Notice that each next permutation here is obtained from the previous one by an extra flip.

(5) The permutation $\binom{1234}{4321}$ inverses all the 6 pairs of indices and has therefore length $l=6$. Thus the elementary product $a_{14}a_{23}a_{32}a_{41}$ occurs with the sign $\varepsilon=(-1)^6=+1$ in the definition of 4×4-determinants.

(6) Since inverse permutations have the same parity, the definition of determinants can be rewritten "by columns:"

$$\det A=\sum_{\sigma}\varepsilon(\sigma)a_{\sigma(1)1}...a_{\sigma(n)n}.$$

Indeed, each summand in this formula is equal to the summand in the original definition corresponding to the permutation σ^{-1}, and *vice versa.* Namely, reordering the factors $a_{\sigma(1)1}...a_{\sigma(n)n}$, so that $\sigma(1),\dots,\sigma(n)$ increase monotonically, yields $a_{1\sigma^{-1}(1)}...a_{n\sigma^{-1}(n)}$.

EXERCISES

206. List all the 24 permutations of $\{1,2,3,4\}$, find the length and the sign of each of them. ↯

207. Find the length of the following permutation:

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & 2k \\ 1 & 3 & \dots & 2k-1 & 2 & 4 & \dots & 2k \end{pmatrix}. \qquad \checkmark$$

208. Find the maximal possible length of permutations of $\{1,...,n\}$. ↯

209. Find the length of a permutation $\begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix}$ given the length l of the permutation $\begin{pmatrix} 1 & \dots & n \\ i_n & \dots & i_1 \end{pmatrix}$. ✓

210. Prove that inverse permutations have the same length. ↯

211. Compare the parities of permutations of the letters *a,g,h,i,l,m,o,r,t* in the words *logarithm* and *algorithm.* ↯

212. Prove that the identity permutations are the only ones of length 0.

213. Find all permutations of length 1. ✓

214. * Show that every permutation σ can be written as the product of $l(\sigma)$ transpositions of nearby indices. ↯

215. * Represent the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$ as a composition of as few transpositions as possible. ✓

216. Do the products $a_{13}a_{24}a_{53}a_{41}a_{35}$ and $a_{21}a_{13}a_{34}a_{55}a_{42}$ occur in the defining formula for determinants of size 5? ✓

217. Find the signs of the elementary products $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$ and $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$ in the definition of determinants of size 6 by computing the numbers of inverted pairs of indices. ✓

Properties of determinants

(i) ***Transposed matrices have equal determinants:***

$$\det A^t = \det A.$$

This follows from the last Example. Below, we will think of an $n \times n$ matrix as an array $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$ of its n columns of size n (vectors from $\mathbb{C}^n$ if you wish) and formulate all further properties of determinants in terms of columns. The same properties hold true for rows, since the transposition of A changes columns into rows without changing the determinant.

(ii) ***Interchanging any two columns changes the sign of the determinant:***

$$\det[..., \mathbf{a}_j, ..., \mathbf{a}_i, ...] = -\det[..., \mathbf{a}_i, ..., \mathbf{a}_j, ...].$$

Indeed, the operation replaces each permutation in the definition of determinants by its composition with the transposition of the indices i and j. Thus changes the parity of the permutation, and thus reverses the sign of each summand.

Rephrasing this property, one says that the determinant, considered as a function of n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ is **totally anti-symmetric**, i.e. changes the sign under every odd permutation of the vectors, and stays invariant under even. It implies that *a matrix with two equal columns has zero determinant.* It also allows one to formulate further column properties of determinants referring to the 1st column only, since the properties of all columns are alike.

(iii) ***Multiplication of a column by a number multiplies the determinant by this number:***

$$\det[\lambda\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n] = \lambda \det[\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n].$$

Indeed, this operation simply multiplies each of the $n!$ elementary products by the factor of λ.

This property shows that *a matrix with a zero column has zero determinant.*

(iv) ***The determinant function is*** **additive** ***with respect to each column:***

$$\det[\mathbf{a}_1' + \mathbf{a}_1'', \mathbf{a}_2, ..., \mathbf{a}_n] = \det[\mathbf{a}_1', \mathbf{a}_2, ..., \mathbf{a}_n] + \det[\mathbf{a}_1'', \mathbf{a}_2, ..., \mathbf{a}_n].$$

Indeed, each elementary product contains exactly one factor picked from the 1-st column and thus splits into the sum of two elementary products $a'_{\sigma(1)1}a_{\sigma(2)2}...a_{\sigma(n)n}$ and $a''_{\sigma(1)1}a_{\sigma(2)2}...a_{\sigma(n)n}$. Summing up over all permutations yields the sum of two determinants on the right hand side of the formula.

The properties (*iv*) and (*iii*) together mean that *the determinant function is linear with respect to each column* separately. Together with the property (*ii*), they show that ***adding a multiple of one column to another one does not change the determinant of the matrix***. Indeed,

$$|\mathbf{a}_1 + \lambda\mathbf{a}_2, \mathbf{a}_2, ...| = |\mathbf{a}_1, \mathbf{a}_2, ...| + \lambda\, |\mathbf{a}_2, \mathbf{a}_2, ...| = |\mathbf{a}_1, \mathbf{a}_2, ...|\,,$$

since the second summand has two equal columns.

The determinant function shares all the above properties with the identically zero function. The following property shows that these functions do not coincide.

(v) $\det I = 1$.

Indeed, since all off-diagonal entries of the identity matrix are zeroes, the only elementary product in the definition of $\det A$ that survives is $a_{11}...a_{nn} = 1$.

The same argument shows that *the determinant of any diagonal matrix equals the product of the diagonal entries.* It is not hard to generalize the argument in order to see that the determinant of any upper or lower triangular matrix is equal to the product of the diagonal entries. One can also deduce this from the following factorization property valid for block triangular matrices.

Consider an $n \times n$-matrix $\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ subdivided into four **blocks** A, B, C, D of sizes $m \times m$, $m \times l$, $l \times m$ and $l \times l$ respectively (where of course $m + l = n$). We will call such a matrix **block triangular** if C or B is the zero matrix 0. We claim that

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \;\det D.$$

Indeed, consider a permutation σ of $\{1, ..., n\}$ which sends at least one of the indices $\{1, ..., m\}$ to the other part of the set, $\{m+1, ..., m+l\}$. Then σ must send at least one of $\{m+1, ..., m+l\}$ back to $\{1, ..., m\}$. This means that every elementary product in our $n \times n$-determinant which contains a factor from B must also contain a factor from C, and hence vanish, if $C = 0$. Thus only the permutations σ which permute $\{1, ..., m\}$ separately from $\{m + 1, ..., m + l\}$ contribute to the determinant in question. Elementary products corresponding to such permutations factor into elementary products from $\det A$ and $\det D$ and eventually add up to the product $\det A \det D$.

Of course, the same holds true if $B = 0$ instead of $C = 0$.

We will use the factorization formula in the 1st proof of the following fundamental property of determinants.

EXERCISES

218. Compute the determinants

$$\begin{vmatrix} 13247 & 13347 \\ 28469 & 28569 \end{vmatrix}, \quad \begin{vmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{vmatrix}. \qquad \checkmark$$

219. The numbers 195, 247, and 403 are divisible by 13. Prove that the following determinant is also divisible by 13: $\begin{vmatrix} 1 & 9 & 5 \\ 2 & 4 & 7 \\ 4 & 0 & 3 \end{vmatrix}$. ↯

220. An office and home phone numbers are written as 7×1-matrix O and 1×7-matrix H respectively. Compute $\det(OH)$. ✓

221. How does a determinant change if all of its n columns are rewritten in the opposite order? ✓

222.* Solve the equation $\begin{vmatrix} 1 & x & x^2 & ... & x^n \\ 1 & a_1 & a_1^2 & ... & a_1^n \\ 1 & a_2 & a_2^2 & ... & a_2^n \\ & & & ... & \\ 1 & a_n & a_n^2 & ... & a_n^n \end{vmatrix} = 0$, where all $a_1, ..., a_n$ are given distinct numbers. ✓

223. Prove that an anti-symmetric matrix of size n has zero determinant if n is odd. ↯

Multiplicativity

`Theorem.` ***The determinant is multiplicative with respect to matrix products: for arbitrary*** $n \times n$***-matrices*** A ***and*** B***,***

$$\det(AB) = (\det A)(\det B).$$

We give two proofs: one *ad hoc*, the other more conceptual.

`Proof I.` Consider the auxiliary $2n \times 2n$ matrix $\begin{bmatrix} A & 0 \\ -I & B \end{bmatrix}$ with the determinant equal to the product $(\det A)(\det B)$ according to the factorization formula. We begin to change the matrix by adding to the last n columns linear combinations of the first n columns with such coefficients that the submatrix B is eventually replaced by zero submatrix. Thus, in order to kill the entry b_{kj} we must add the b_{kj}-multiple of the k-th column to the $n+j$-th column. According to the properties of determinants (see (iv)) these operations do not change the determinant but transform the matrix to the form $\begin{bmatrix} A & C \\ -I & 0 \end{bmatrix}$. We ask the reader to check that the entry c_{ij} of the submatrix C in the upper right corner equals $a_{i1}b_{1j} + ... + a_{in}b_{nj}$ so that $C = AB$ is the matrix product! Now, interchanging the i-th and $n+i$-th columns, $i = 1, ..., n$, we change the determinant by the factor of $(-1)^n$ and transform the matrix to the form $\begin{bmatrix} C & A \\ 0 & -I \end{bmatrix}$. The factorization formula applies again and yields $\det C \det(-I)$. We conclude that $\det C = \det A \det B$ since $\det(-I) = (-1)^n$ compensates for the previous factor $(-1)^n$. □

`Proof II.` We will first show that the properties (i – v) completely characterize $\det[\mathbf{v}_1, \dots, \mathbf{v}_n]$ as a function of n columns $\mathbf{v}_i$ of size n.

Indeed, consider a function f, which to n columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, associates a number $f(\mathbf{v}_1, \dots, \mathbf{v}_n)$. Suppose that f is *linear* with respect to each column. Let $\mathbf{e}_i$ denote the ith column of the identity matrix. Since $\mathbf{v}_1 = \sum_{i=1}^n v_{i1}\mathbf{e}_i$, we have:

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \sum_{i=1}^{n} v_{i1} f(\mathbf{e}_i, \mathbf{v}_2, \dots, \mathbf{v}_n).$$

Using linearity with respect to the 2nd column $\mathbf{v}_2 = \sum_{j=1}^n v_{j2}\mathbf{e}_j$, we similarly obtain:

$$f(\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_{i1} v_{j2} f(\mathbf{e}_i, \mathbf{e}_j, \mathbf{v}_3, \dots, \mathbf{v}_n).$$

Proceeding the same way with all columns, we get:

$$f(\mathbf{v}_1, \dots, \mathbf{v}_n) = \sum_{i_1, \dots, i_n} v_{i_1 1} \cdots v_{i_n n} f(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}).$$

Thus, f is determined by its values $f(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n})$ on strings of n basis vectors.

Let us assume now that f is *totally anti-symmetric*. Then, if any two of the indices $i_1, \dots, i_n$ coincide, we have: $f(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = 0$. All other coefficients correspond to *permutations* $\sigma = \begin{pmatrix} 1 & \dots & n \\ i_1 & \dots & i_n \end{pmatrix}$ of the indices $(1, \dots, n)$, and hence satisfy:

$$f(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = \varepsilon(\sigma) f(\mathbf{e}_1, \dots, \mathbf{e}_n).$$

Therefore, we find:

$$\begin{aligned} f(\mathbf{v}_1, \dots, \mathbf{v}_n) &= \sum_{\sigma} v_{\sigma(1)1} \dots v_{\sigma(n)n} \varepsilon(\sigma) f(\mathbf{e}_1, \dots, \mathbf{e}_n), \\ &= f(\mathbf{e}_1, \dots, \mathbf{e}_n) \det[\mathbf{v}_1, \dots, \mathbf{v}_n]. \end{aligned}$$

Thus, we have established:

Proposition 1. ***Every totally anti-symmetric function of*** n ***coordinate vectors of size*** n ***which is linear in each of them is proportional to the determinant function.***

Next, given an $n \times n$ matrix C, put

$$f(\mathbf{v}_1, \dots, \mathbf{v}_n) := \det[C\mathbf{v}_1, \dots, C\mathbf{v}_n].$$

Obviously, the function f is totally anti-symmetric in all $\mathbf{v}_i$ (since det is). Multiplication by C is linear:

$$C(\lambda \mathbf{u} + \mu \mathbf{v}) = \lambda C\mathbf{u} + \mu C\mathbf{v} \quad \text{for all } \mathbf{u}, \mathbf{v} \text{ and } \lambda, \mu.$$

Therefore, f is linear with respect to each $\mathbf{v}_i$ (as composition of two linear operations). By the previous result, f is proportional to det.

Since $C\mathbf{e}_i$ are columns of C, we conclude that the coefficient of proportionality $f(\mathbf{e}_1, \dots, \mathbf{e}_n) = \det C$. Thus, we have found the following interpretation of $\det C$.

`Proposition 2.` $\det C$ ***is the factor by which the determinant function of*** n ***vectors*** $\mathbf{v}_i$ ***is multiplied when the vectors are replaced with*** $C\mathbf{v}_i$***.***

Now our theorem follows from the fact that when $C = AB$, the substitution $\mathbf{v} \mapsto C\mathbf{v}$ is the composition $\mathbf{v} \mapsto A\mathbf{v} \mapsto AB\mathbf{v}$ of consecutive substitutions defined by A and B. Under the action of A, the function det is multiplied by the factor $\det A$, then under the action of B by another factor $\det B$. But the resulting factor $(\det A)(\det B)$ must be equal to $\det C$. □

`Corollary.` ***If*** A ***is invertible, then*** $\det A$ ***is invertible.***

Indeed, $(\det A)(\det A^{-1}) = \det I = 1$, and hence $\det A^{-1}$ is reciprocal to $\det A$. The converse statement: that matrices with invertible determinants are invertible, is also true due to the explicit formula for the inverse matrix, described in the next section.

`Remark.` Of course, a real or complex number $\det A$ is invertible whenever $\det A \neq 0$. Yet over the integers $\mathbb{Z}$ this is not the case: the only invertible integers are ± 1. The above formulation, and several similar formulations that follow, which refer to invertibility of determinants, are preferable as they are more general.

EXERCISES

224. How do similarity transformations of a given matrix affect its determinant? ✓

225. Prove that the sign of the determinant of the coefficient matrix of a real quadratic form does not depend on the coordinate system. ↯

The Cofactor Theorem

In the determinant formula for an $n \times n$-matrix A each elementary product $\pm a_{1\sigma(1)} \dots$ begins with one of the entries $a_{11}, \dots, a_{1n}$ of the first row. The sum of all terms containing a_{11} in the 1-st place is the product of a_{11} with the determinant of the $(n-1) \times (n-1)$-matrix obtained from A by crossing out the 1-st row and the 1-st column. Similarly, the sum of all terms containing a_{12} in the 1-st place looks like the product of a_{12} with the determinant obtained by crossing out the 1-st row and the 2-nd column of A. In fact it differs by the factor of -1 from this product, since switching the columns 1 and 2 changes signs of all terms in the determinant formula and interchanges the roles of a_{11} and a_{12}. Proceeding in this way with $a_{13}, ..., a_{1n}$ we arrive at the **cofactor expansion** formula for $\det A$ which can be stated as follows.

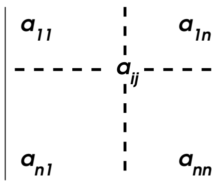

Figure 29

$i \backslash j$	1	2	3	4	5
1	+	−	+	−	+
2	−	+	−	+	−
3	+	−	+	−	+
4	−	+	−	+	−
5	+	−	+	−	+

Figure 30

The determinant of the $(n-1) \times (n-1)$-matrix obtained from A by crossing out the row i and column j is called the (ij)**-minor** of A (Figure 29). Denote it by M_{ij}. The (ij)**-cofactor** A_{ij} of the matrix A is the number that differs from the minor M_{ij} by a factor ± 1:

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

The chess-board of the signs $(-1)^{i+j}$ is shown on Figure 30. With these notations, the cofactor expansion formula reads:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + ... + a_{1n}A_{1n}.$$

Example.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Using the properties (i) and (ii) of determinants we can adjust the cofactor expansion to the i-th row or j-th column:

$$\det A = a_{i1}A_{i1} + ... + a_{in}A_{in} = a_{1j}A_{1j} + ... + a_{nj}A_{nj}, \ \ i,j = 1, ..., n.$$

These formulas reduce evaluation of $n \times n$-determinants to that of $(n-1) \times (n-1)$-determinants and can be useful in recursive computations.

Furthermore, we claim that applying the cofactor formula to the entries of the i-th row but picking the cofactors of another row we get the zero sum:

$$a_{i1}A_{j1} + ... + a_{in}A_{jn} = 0 \text{ if } i \neq j.$$

Indeed, construct a new matrix $\tilde{A}$ replacing the j-th row by a copy of the i-th row. This forgery does not change the cofactors $A_{j1}, ..., A_{jn}$ (since the j-th row is crossed out anyway) and yields the cofactor expansion $a_{i1}A_{j1}+...+a_{in}A_{jn}$ for $\det \tilde{A}$. But $\tilde{A}$ has two identical rows and hence $\det \tilde{A} = 0$. The same arguments applied to the columns yield the dual statement:

$$a_{1i}A_{1j} + ... + a_{ni}A_{nj} = 0 \text{ if } i \neq j.$$

All the above formulas can be summarized in a single matrix identity. Introduce the $n \times n$-matrix $\operatorname{adj}(A)$, called **adjugate** to A, by placing the cofactor A_{ij} on the intersection of j-th row and i-th column. In other words, each a_{ij} is replaced with the corresponding cofactor A_{ij}, and then the resulting matrix is transposed:

$$\operatorname{adj} \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & a_{ij} & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{n1} \\ \dots & A_{ji} & \dots \\ A_{1n} & \dots & A_{nn} \end{bmatrix}.$$

`Theorem.` $A \operatorname{adj}(A) = (\det A)\, I = \operatorname{adj}(A)\, A.$

`Corollary.` ***If*** $\det A$ ***is invertible then*** A ***is invertible, and***

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A).$$

`Example.` If $ad - bc \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

EXERCISES

226. Prove that the adjugate matrix of an upper (lower) triangular matrix is upper (lower) triangular.

227. Which triangular matrices are invertible?

228. Compute the determinants: ($*$ is a wild card):

$$(a) \begin{vmatrix} * & * & * & a_n \\ * & * & \dots & 0 \\ * & a_2 & 0 & \dots \\ a_1 & 0 & \dots & 0 \end{vmatrix}, \quad (b) \begin{vmatrix} * & * & a & b \\ * & * & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{vmatrix}. \qquad \checkmark$$

229. Compute determinants using cofactor expansions:

$$(a) \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{vmatrix}, \quad (b) \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}. \qquad \checkmark$$

230. Compute the inverses of matrices using the Cofactor Theorem:

$$(a)\ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \quad (b)\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \checkmark$$

231. Solve the systems of linear equations $A\mathbf{x} = \mathbf{b}$ where A is one of the matrices of the previous exercise, and $\mathbf{b} = [1, 0, 1]^t$. ✓

232. Compute

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

233. Express $\det(\mathrm{adj}(A))$ of the adjugate matrix via $\det A$. ✓

234. Which integer matrices have integer inverses? ✓

Cramer's Rule

This is an application of the Cofactor Theorem to systems of linear equations. Consider a system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

of n linear equations with n unknowns $(x_1, \ldots, x_n)$. It can be written in the matrix form

$$A\mathbf{x} = \mathbf{b},$$

where A is the $n \times n$-matrix of the coefficients a_{ij}, $\mathbf{b} = [b_1, \ldots, b_n]^t$ is the column of the right hand sides, and $\mathbf{x}$ is the column of unknowns. In the following Corollary, $\mathbf{a}_i$ denote columns of A.

`Corollary.` ***If*** $\det A$ ***is invertible then the system of linear equations*** $A\mathbf{x} = \mathbf{b}$ ***has a unique solution given by the formulas:***

$$x_1 = \frac{\det[\mathbf{b}, \mathbf{a}_2, ..., \mathbf{a}_n]}{\det[\mathbf{a}_1, ..., \mathbf{a}_n]}, \ \ldots\ , x_n = \frac{\det[\mathbf{a}_1, ..., \mathbf{a}_{n-1}, \mathbf{b}]}{\det[\mathbf{a}_1, ..., \mathbf{a}_n]}.$$

Indeed, when $\det A \neq 0$, the matrix A is invertible. Multiplying the matrix equation $A\mathbf{x} = \mathbf{b}$ by A^{-1} on the left, we find:

$\mathbf{x} = A^{-1}\mathbf{b}$. Thus the solution is unique, and $x_i = (\det A)^{-1}(A_{1i}b_1 + ... + A_{ni}b_n)$ according to the cofactor formula for the inverse matrix. But the sum $b_1A_{1i} + ... + b_nA_{ni}$ is the cofactor expansion for $\det[\mathbf{a}_1, ..., \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, ..., \mathbf{a}_n]$ with respect to the i-th column.

`Example.` Suppose that $a_{11}a_{22} \neq a_{12}a_{21}$. Then the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_2 + a_{22}x_2 &= b_2 \end{aligned}$$

has a unique solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

EXERCISES

235. Solve systems of equations using Cramer's rule:

$$(a) \quad \begin{array}{rcl} 2x_1 - x_2 - x_3 & = & 4 \\ 3x_1 + 4x_2 - 2x_3 & = & 11 \\ 3x_1 - 2x_2 + 4x_3 & = & 11 \end{array}, \quad (b) \quad \begin{array}{rcl} x_1 + 2x_2 + 4x_3 & = & 31 \\ 5x_1 + x_2 + 2x_3 & = & 29 \\ 3x_1 - x_2 + x_3 & = & 10 \end{array}. \qquad \checkmark$$

Three cool formulas

We collect here some useful generalizations of previous results.

A. We don't know of any reasonable generalization of determinants to the situation when matrix entries do *not* commute. However the following generalization of the formula $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ is instrumental in some non-commutative applications.[10]

In the block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$***, assume that*** D^{-1} ***exists. Then*** $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\det D$***.***

`Proof:` $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & B \\ 0 & D \end{bmatrix}$.

[10] Notably in the definition of *Berezinian* in super-mathematics [10].

B. Laplace's formula[11] below generalizes cofactor expansions.

By a **multi-index** I of length $|I| = k$ we mean an increasing sequence $i_1 < \cdots < i_k$ of k indices from the set $\{1, \ldots, n\}$. Given and $n \times n$-matrix A and two multi-indices I, J of the same length k, we define the (IJ)**-minor** M_{IJ} of A as the determinant of the $k \times k$-matrix formed by the entries $a_{\mathbf{i}_\alpha j_\beta}$ of A located at the intersections of the rows $i_1, \ldots, i_k$ with columns $j_1, \ldots, j_k$ (see Figure 31). Also, denote by $\bar{I}$ the multi-index **complementary** to I, i.e. formed by those $n - k$ indices from $\{1, \ldots, n\}$ which are *not* contained in I.

For each multi-index $I = (i_1, \ldots, i_k)$***, the following* cofactor expansion with respect to rows** $i_1, \ldots, i_k$ **holds true:**

$$\det A = \sum_{J:|J|=k} (-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k} M_{IJ} M_{\bar{I}\bar{J}},$$

where the sum is taken over all multi-indices $J = (j_1, \ldots, j_k)$ ***of length*** k***.***

Similarly, one can similarly write Laplace's cofactor expansion formula with respect to given k columns.

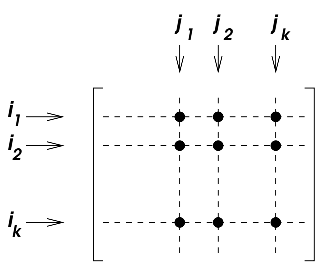

Figure 31

`Example.` Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ be 8 vectors on the plane. Then $\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{vmatrix} = |\mathbf{a}_1\ \mathbf{a}_2||\mathbf{b}_3\ \mathbf{b}_4| - |\mathbf{a}_1\ \mathbf{a}_3||\mathbf{b}_2\ \mathbf{b}_4|$ $+ |\mathbf{a}_1\ \mathbf{a}_4||\mathbf{b}_2\ \mathbf{b}_3| + |\mathbf{a}_2\ \mathbf{a}_3||\mathbf{b}_1\ \mathbf{b}_4| - |\mathbf{a}_2\ \mathbf{a}_4||\mathbf{b}_1\ \mathbf{b}_3| + |\mathbf{a}_3\ \mathbf{a}_4||\mathbf{b}_1\ \mathbf{b}_2|$.

In the proof of Laplace's formula, it suffices to assume that it is written with respect to the *first* k rows, i.e. that $I = (1, \ldots, k)$. Indeed, interchanging them with the rows $i_1 < \cdots < i_k$ takes $(i_1 - 1) + (i_2 - 2) + \cdots + (i_k - k)$ transpositions, which is accounted for by the sign $(-1)^{i_1+\cdots+i_k}$ in the formula.

[11] After Pierre-Simon **Laplace** (1749–1827).

Next, multiplying out $M_{IJ}M_{\bar I\bar J}$, we find $k!(n-k)!$ elementary products of the form:

$$\pm a_{1,j_{\alpha_1}}\cdots a_{k,j_{\alpha_k}}\ a_{k+1,\bar j_{\beta_1}}\cdots a_{n,\bar j_{\beta_{n-k}}},$$

where $\alpha = \begin{pmatrix} 1 & \dots & k \\ \alpha_1 & \dots & \alpha_k \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & \dots & n-k \\ \beta_1 & \dots & \beta_{n-k} \end{pmatrix}$ are permutations, and $j_{\alpha_\mu} \in J$, $\bar j_{\beta_\nu} \in \bar J$. It is clear that the total sum over multi-indices I contains each elementary product from $\det A$, and does it exactly once. Thus, to finish the proof, we need to compare the signs.

The sign $\pm$ in the above formula is equal to $\varepsilon(\alpha)\varepsilon(\beta)$, the product of the signs of the permutations α and β. The sign of this elementary product in the definition of $\det A$ is equal to the sign of the permutation $\begin{pmatrix} 1 & \dots & k & k+1 & \dots & n \\ j_{\alpha_1} & \dots & j_{\alpha_k} & \bar j_{\beta_1} & \dots & \bar j_{\beta_{n-k}} \end{pmatrix}$ on the set $J \cup \bar J = \{1,\dots,n\}$. Reordering separately the first k and last $n-k$ indices in the increasing order changes the sign of the permutation by $\varepsilon(\alpha)\varepsilon(\beta)$. Therefore the signs of all summands of $\det A$ which occur in $M_{IJ}M_{\bar I\bar J}$ are *coherent*. It remains to find the total sign with which $M_{IJ}M_{\bar I\bar J}$ occurs in $\det A$, by computing the sign of the permutation $\sigma := \begin{pmatrix} 1 & \dots & k & k+1 & \dots & n \\ j_1 & \dots & j_k & \bar j_1 & \dots & \bar j_{n-k} \end{pmatrix}$, where $j_1 < \cdots j_k$ and $\bar j_1 < \cdots < \bar j_{n-k}$.

Starting with the identity permutation $(1, 2\dots, j_1, \dots, j_2, \dots, n)$, it takes $j_1 - 1$ transpositions of nearby indices to move j_1 to the 1st place. Then it takes $j_2 - 2$ such transpositions to move j_2 to the 2nd place. Continuing this way, we find that

$$\varepsilon(\sigma) = (-1)^{(j_1-1)+\cdots+(j_k-k)} = (-1)^{1+\cdots+k+j_1+\cdots+\mathbf{j}_k}.$$

This agrees with Laplace's formula, since $I = \{1,\dots,k\}$. $\square$.

C. Let A and B be $k\times n$ and $n\times k$ matrices (think of $k<n$). For each multi-index $I = (i_1,\dots,i_k)$, denote by A_I and B_I the $k\times k$-matrices formed by respectively: columns of A and rows of B with the indices $i_1,\dots,i_k$.

The determinant of the $k\times k$***-matrix*** AB ***is given by the following*** **Binet–Cauchy formula**:[12]

$$\det AB = \sum_I (\det A_I)(\det B_I).$$

[12] After Jacques **Binet** (1786–1856) and Augustin Louis **Cauchy** (1789–1857).

Note that when $k = n$, this turns into the multiplicative property of determinants: $\det(AB) = (\det A)(\det B)$. Our second proof of it can be generalized to establish the formula of Binet–Cauchy. Namely, let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote columns of A. Then the jth column of $C = AB$ is the linear combination: $\mathbf{c}_j = \mathbf{a}_1 b_{1j} + \cdots + \mathbf{a}_n b_{nj}$. Using linearity in each $\mathbf{c}_j$, we find:

$$\det[\mathbf{c}_1, \ldots, \mathbf{c}_k] = \sum_{1 \le i_1, \ldots, i_k \le n} \det[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}] b_{i_1 1} \cdots b_{i_k k}.$$

If any two of the indices i_α coincide, $\det[\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}] = 0$. Thus the sum is effectively taken over all *permutations* $\begin{pmatrix} 1 & \ldots & k \\ i_1 & \ldots & i_k \end{pmatrix}$ on the *set*[13] $\{i_1, \ldots, i_k\}$. Reordering the columns $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}$ in the increasing order of the indices (and paying the "fees" ± 1 according to the parities of the permutations) we obtain the sum over all multi-indices of length k:

$$\sum_{i'_1 < \cdots < i'_k} \det[\mathbf{a}_{i'_1}, \ldots, \mathbf{a}_{i'_k}] \sum_{\sigma} \varepsilon(\sigma) b_{i_1 1} \cdots b_{i_k k}.$$

The sum on the right is taken over permutations $\sigma = \begin{pmatrix} i'_1 & \ldots & i'_k \\ i_1 & \ldots & i_k \end{pmatrix}$. It is equal to $\det B_I$, where $I = (i'_1, \ldots, i'_k)$. □

`Corollary 1.` *If* $k > n$, $\det AB = 0$.

This is because no multi-indices of length $k > n$ can be formed from $\{1, \ldots, n\}$. In the other extreme case $k = 1$, Binet–Cauchy's formula turns into the expression $\mathbf{u}^t \mathbf{v} = \sum u_i v_i$ for the dot product of coordinate vectors.

`Corollary 2.` $\det AA^t = \sum_I (\det A_I)^2$.

EXERCISES

236.* Compute determinants:

$$(a) \begin{vmatrix} 0 & x_1 & x_2 & \ldots & x_n \\ x_1 & 1 & 0 & \ldots & 0 \\ x_2 & 0 & 1 & \ldots & 0 \\ . & . & . & . & . \\ x_n & 0 & \ldots & 0 & 1 \end{vmatrix}, \quad (b) \begin{vmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & d \end{vmatrix} \quad ϟ \checkmark.$$

[13] Remember that in a set, elements are unordered!

237.* Let P_{ij}, $1 \le i < j \le 4$, denote the 2×2-minor of a 2×4-matrix formed by the columns i and j. Prove the following **Plücker identity**[14]

$$P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0. \qquad ϟ''$$

238. The **cross product** of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is defined by

$$\mathbf{x} \times \mathbf{y} := \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, \begin{vmatrix} x_3 & x_1 \\ y_3 & y_1 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right).$$

Prove that the length $|\mathbf{x}\times\mathbf{y}| = \sqrt{|\mathbf{x}|^2|\mathbf{y}|^2 - \langle \mathbf{x}, \mathbf{y}\rangle^2}$. ϟ

239.* Prove that $a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\dots + \cfrac{1}{a_1 + \cfrac{1}{a_0}}}} = \dfrac{\Delta_n}{\Delta_{n-1}}$,

where $\Delta_n = \begin{vmatrix} a_0 & 1 & 0 & \dots & 0 \\ -1 & a_1 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & \dots & -1 & a_{n-1} & 1 \\ 0 & \dots & 0 & -1 & a_n \end{vmatrix}$. ϟ

240.* Compute: $\begin{vmatrix} \lambda & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \dots & 0 \\ . & . & . & . & . \\ 0 & \dots & 0 & \lambda & -1 \\ a_n & a_{n-1} & \dots & a_2 & \lambda + a_1 \end{vmatrix}$. ✓

241.* Compute: $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \binom{2}{1} & \binom{3}{1} & \dots & \binom{n}{1} \\ 1 & \binom{3}{2} & \binom{4}{2} & \dots & \binom{n+1}{2} \\ . & . & . & . & . \\ 1 & \binom{n}{n-1} & \binom{n+1}{n-1} & \dots & \binom{2n-2}{n-1} \end{vmatrix}$. ϟ ✓

242.* Prove **Vandermonde's identity**[15]

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ . & . & . & . & . \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i). \qquad ϟ$$

243.* Compute: $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^3 & 3^3 & \dots & n^3 \\ . & . & . & . & . \\ 1 & 2^{2n-1} & 3^{2n-1} & \dots & n^{2n-1} \end{vmatrix}$. ϟ ✓

[14] After Julius **Plücker** (1801–1868).
[15] After Alexandre-Theóphile **Vandermonde** (1735–1796).

3 The Inertia Theorem

We study here the classification of quadratic forms and some generalizations of this problem. The answer actually depends on properties of the field of scalars. This section focuses on the cases $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, while some other cases are delegated to the next section. We begin, however, with a key argument that remains valid in general. [4]

Orthogonal bases

In section "Matrices" of Chapter 2 we established a one-to-one correspondence between symmetric bilinear forms and quadratic forms. To recall, a **symmetric bilinear form** on a vector space $\mathcal{V}$ is a function $Q : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$, $(\mathbf{x}, \mathbf{y}) \mapsto Q(\mathbf{x}, \mathbf{y})$, which is linear in each vector variable $\mathbf{x}$ and $\mathbf{y}$, and symmetric, i.e. $Q(\mathbf{y}, \mathbf{x}) = Q(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$. Taking the arguments in a symmetric bilinear form equal to each other, one obtains the corresponding quadratic form, which we will denote by the same letter: $Q : \mathcal{V} \to \mathbb{K}$. Thus, $Q(\mathbf{x}) := Q(\mathbf{x}, \mathbf{x})$. (This should not cause confusion: whenever there are two arguments, it is the bilinear form, and when there is only one, it is the corresponding quadratic form.) The symmetric bilinear form is reconstructed from the corresponding quadratic form as

$$Q(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\left[Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})\right].$$

In coordinates, if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathcal{V}$, we have

$$Q(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i q_{ij} y_j = \mathbf{x}^t Q \mathbf{y}, \text{ where } q_{ij} = Q(\mathbf{e}_i, \mathbf{e}_j) = q_{ji},$$

and respectively $Q(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i q_{ij} x_j = \mathbf{x}^t Q \mathbf{x}$.

Under a linear change of coordinates $\mathbf{x} = C\mathbf{x}', \mathbf{y} = C\mathbf{y}'$, the symmetric coefficient matrix $Q = [q_{ij}]$ changes according to the transformation rule $Q \mapsto C^t Q C$.

A basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ in the space $\mathcal{V}$ is called **Q-orthogonal** if $Q(\mathbf{f}_i, \mathbf{f}_j) = 0$ for all $i \neq j$, i.e. if the symmetric coefficient matrix of the quadratic form with respect to this basis is diagonal.

`Lemma.` ***Every quadratic form in a finite dimensional vector space has an orthogonal basis.***

[4] More precisely, whenever $\mathbb{K}$ does not contain $\mathbb{Z}_2$, so that $1/2$ exists.

`Proof.` We use induction on the dimension $n = \dim \mathcal{V}$ of the vector space. For $n = 1$ the requirement is empty. Let us construct a Q-orthogonal basis in $\mathcal{V}$ assuming that every quadratic form in space of dimension $n-1$ has an orthogonal basis. If the given quadratic form Q is identically zero, the corresponding symmetric bilinear form is identically zero too, and so any basis is Q-orthogonal. If the quadratic form is not identically zero, then there exists a vector $\mathbf{f}_1$ such that $Q(\mathbf{f}_1) \neq 0$. Let $\mathcal{W}$ be the subspace in $\mathcal{V}$ consisting of all vectors Q-orthogonal to $\mathbf{f}_1$: $\mathcal{W} = \{\mathbf{x} \in \mathcal{V} \mid Q(\mathbf{f}_1, \mathbf{x}) = 0.\}$. This subspace does *not* contain $\mathbf{f}_1$ and is given by 1 linear equation. Thus $\dim \mathcal{W} = n-1$. Let $\{\mathbf{f}_2, \dots, \mathbf{f}_n\}$ be a basis in $\mathcal{W}$ orthogonal with respect to the symmetric bilinear form obtained by restricting Q to this subspace. Such a basis exists by the induction hypothesis. Therefore $Q(\mathbf{f}_i, \mathbf{f}_j) = 0$ for all $1 < i < j$. Besides, $Q(\mathbf{f}_1, \mathbf{f}_i) = 0$ for all $i > 1$, since $\mathbf{f}_i \in \mathcal{W}$. Then $Q(\mathbf{f}_i, \mathbf{f}_j) = 0$ for all $i > j$ by the symmetry of Q. Thus $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ is a Q-orthogonal basis of $\mathcal{V}$. □

`Corollary.` ***For every symmetric*** $n \times n$***-matrix*** Q ***with entries from*** $\mathbb{K}$ ***there exists an invertible matrix*** C ***such that*** $C^t Q C$ ***is diagonal.***

The diagonal entries here are the values $Q(\mathbf{f}_1), \dots, Q(\mathbf{f}_n)$.

Inertia indices

Consider the case $\mathbb{K} = \mathbb{R}$.

Given a quadratic form Q in $\mathbb{R}^n$, we pick a Q-orthogonal basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ and then *rescale* those of the basis vectors for which $Q(\mathbf{f}_i) \neq 0$: $\mathbf{f}_i \mapsto \tilde{\mathbf{f}}_i = |Q(\mathbf{f}_i)|^{-1/2} \mathbf{f}_i$. After such rescaling, the non-zero coefficients $Q(\tilde{\mathbf{f}}_i)$ of the quadratic form will become ± 1. Reordering the basis so that the terms with positive coefficients come first, and negative next, we transform Q to the normal form:

$$Q = X_1^2 + \cdots + X_p^2 - X_{p+1}^2 - \cdots - X_{p+q}^2, \quad p + q \le n.$$

Note that by restricting Q to the subspace $X_{p+1} = \cdots = X_n = 0$ of dimension p we obtain a quadratic form on this subspace which is **positive** (or **positive definite**), i.e takes on positive values everywhere outside the origin.

`Proposition.` ***The numbers*** p ***and*** q ***of positive and negative squares in the normal form are equal to the maximal dimensions of the subspaces in*** $\mathbb{R}^n$ ***where the quadratic form*** Q ***(respectively,*** $-Q$***) is positive.***

`Proof.` The quadratic form $Q = X_1^2 + ... + X_p^2 - X_{p+1}^2 - ... - X_{p+q}^2$ is *non*-positive everywhere on the subspace $\mathcal{W}$ of dimension $n-p$ given by the equations $X_1 = ... = X_p = 0$. Let us show that the existence of a subspace $\mathcal{V}$ of dimension $p+1$ where the quadratic form is positive leads to a contradiction. Indeed, the subspaces $\mathcal{V}$ and $\mathcal{W}$ would intersect in a subspace of dimension at least $(p+1)+(n-p)-n = 1$, containing therefore non-zero vectors $\mathbf{x}$ with $Q(\mathbf{x}) > 0$ and $Q(\mathbf{x}) \leq 0$. Thus, Q is positive on some subspace of dimension p and cannot be positive on any subspace of dimension $> p$. Likewise, $-Q$ is positive on some subspace of dimension q and cannot be positive on any subspace of dimension $> q$. $\square$

The maximal dimensions of positive subspaces of Q and $-Q$ are called respectively **positive** and **negative inertia indices** of a quadratic form in question. By definition, inertia indices of a quadratic form do not depend on the choice of a coordinate system. Our Proposition implies that the normal forms with different pairs of values of p and q are pairwise non-equivalent. This establishes the Inertia Theorem (as stated in Section 4 of Chapter 1).

`Theorem.` ***Every quadratic form in $\mathbb{R}^n$ by a linear change of coordinates can be transformed to exactly one of the normal forms:***

$$X_1^2 + ... + X_p^2 - X_{p+1}^2 - ... - X_{p+q}^2, \quad \textit{where } 0 \leq p+q \leq n.$$

The matrix formulation of the Inertia Theorem reads:

Every real symmetric matrix Q can be transformed to exactly one of the diagonal forms $\begin{bmatrix} I_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ *by transformations of the form $Q \mapsto C^tQC$ defined by invertible real matrices C.*

EXERCISES

303. Find orthogonal bases and the inertia indices of quadratic forms: ✓

$$x_1x_2 + x_2^2, \quad x_1^2 + 4x_1x_2 + 6x_2^2 - 12x_2x_3 + 18x_3^2, \quad x_1x_2 + x_2x_3 + x_3x_1.$$

304. Prove that $Q = \sum_{1 \leq i \leq j \leq n} x_ix_j$ is positive definite.

305. A minor of a square matrix formed by rows and columns with the same indices is called **principal**. Prove that all principal minors of the coefficient matrix of a positive definite quadratic form are positive.

306.* Let $\mathbf{a}_1, \dots, \mathbf{a}_p$ and $\mathbf{b}_1, \dots, \mathbf{b}_q$ be linear forms in $\mathbb{R}^n$, and let $Q(\mathbf{x}) = \mathbf{a}_1^2(\mathbf{x}) + \cdots + \mathbf{a}_p^2(\mathbf{x}) - \mathbf{b}_1^2(\mathbf{x}) - \cdots - \mathbf{b}_q^2(\mathbf{x})$. Prove that the positive and negative inertia indices of Q do not exceed p and q respectively. ↯

Complex quadratic forms

Consider the case $\mathbb{K} = \mathbb{C}$.

Theorem. ***Every quadratic form in $\mathbb{C}^n$ can be transformed by linear changes of coordinates to exactly one of the normal forms:***

$$z_1^2 + \cdots + z_r^2, \quad \textit{where } 0 \le r \le n.$$

Proof. Given a quadratic form Q, pick a Q-orthogonal basis in $\mathbb{C}^n$, order it in such a way that vectors $\mathbf{f}_1, \dots, \mathbf{f}_r$ with $Q(\mathbf{f}_i) \neq 0$ come first, and then rescale these vectors by $\mathbf{f}_i \mapsto Q(\mathbf{f}_i)^{-1/2}\mathbf{f}_i$.

In particular, we have proved that *every complex symmetric matrix Q can be transformed to exactly one of the forms* $\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ *by the transformations of the form $Q \mapsto C^t Q C$ defined by invertible complex matrices C.* As it follows from the Rank Theorem, here $r = \operatorname{rk} Q$, the rank of the coefficient matrix of the quadratic form. This guarantees that the normal forms with different values of r are pairwise non-equivalent, and thus completes the proof. □

To establish the geometrical meaning of r, consider a more general situation.

Given a quadratic form Q on a $\mathbb{K}$-vector space $\mathcal{V}$, its **kernel** is defined as the subspace of $\mathcal{V}$ consisting of all vectors which are Q-orthogonal to all vectors from $\mathcal{V}$:

$$\operatorname{Ker} Q := \{\mathbf{z} \in \mathcal{V} | Q(\mathbf{z}, \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in \mathcal{V}.\}$$

Note that the values $Q(\mathbf{x}, \mathbf{y})$ do not change when a vector from the kernel is added to either of $\mathbf{x}$ and $\mathbf{y}$.[5]

The **rank** of a quadratic form Q on $\mathbb{K}^n$ is defined as the codimension of $\operatorname{Ker} Q$. For example, the quadratic form $z_1^2 + \cdots + z_r^2$ on $\mathbb{K}^n$ corresponds to the symmetric bilinear form $x_1y_1 + \cdots + x_ry_r$, and has the kernel of codimension r defined by the equations $z_1 = \cdots = z_r = 0$.

[5] As a result, the symmetric bilinear form Q descends to the *quotient space* $\mathcal{V}/\operatorname{Ker} Q$ (see Supplement D).

Conics

The set of all solutions to one polynomial equation in $\mathbb{K}^n$:

$$F(x_1, \dots, x_n) = 0$$

is called a **hypersurface**. When the polynomial F does not depend on one of the variables (say, x_n), the equation $F(x_1, \dots, x_{n-1}) = 0$ defines a hypersurface in $\mathbb{K}^{n-1}$. Then the solution set in $\mathbb{K}^n$ is called a **cylinder**, since it is the Cartesian product of the hypersurface in $\mathbb{K}^{n-1}$ and the line of arbitrary values of x_n.

Hypersurfaces defined by polynomial equations of degree 2 are often referred to as **conics** — a name reminiscent of conic sections, which are "hypersurfaces" in $\mathbb{K}^2$. The following application of the Inertia Theorem allows one to classify all conics in $\mathbb{R}^n$ up to **equivalence** defined by compositions of translations with invertible linear transformations.

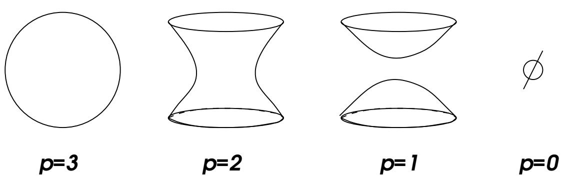

Figure 38

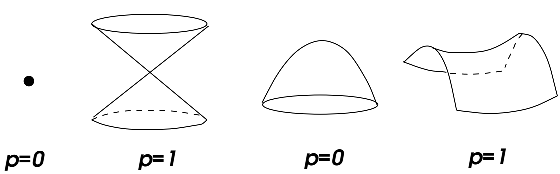

Figure 39

Theorem. ***Every conic in*** $\mathbb{R}^n$ ***is equivalent to either the cylinder over a conic in*** $\mathbb{R}^{n-1}$***, or to one of the conics:***

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2 = 1, \quad 0 \le p \le n,$$
$$x_1^2 + \cdots + x_p^2 = x_{p+1}^2 + \cdots + x_n^2, \quad 0 \le p \le n/2,$$
$$x_n = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{n-1}^2, \quad 0 \le p \le (n-1)/2,$$

known as hyperboloids, cones, and paraboloids respectively.

For $n = 3$, all types of "hyperboloids" (of which the first type contains spheres and ellipsoids) are shown in Figure 38, and cones and paraboloids in Figure 39.

`Proof.` Given a degree 2 polynomial $F = Q(\mathbf{x}) + \mathbf{a}(\mathbf{x}) + c$, where Q is a non-zero quadratic form, $\mathbf{a}$ a linear form, and c a constant, we can apply a linear change of coordinates to transform Q to the form $\pm x_1^2 \pm \cdots \pm x_r^2$, where $r \leq n$, and then use the completion of squares in the variables $x_1, \dots, x_r$ to make the remaining linear form independent of $x_1, \dots, x_r$. When $r = n$, the resulting equations $\pm x_1^2 \pm \cdots \pm x_n^2 = C$ (where C is a new constant) define hyperboloids (when $C \neq 0$), or cones (when $C = 0$). When $r < n$, we can take the remaining *linear* part of the function F (together with the constant) for a new, $r+1$-st coordinate, provided that this linear part is non-constant. When $r = n - 1$, we obtain the equations of paraboloids. When $r < n-1$, or if $r = n-1$, but the linear function was constant, the function F, written in new coordinates, does not depend on the last of them, and this defines the cylinder over a conic in $\mathbb{R}^{n-1}$. □

Classification of conics in $\mathbb{C}^n$ is obtained in the same way, but the answer looks simpler, since there are no signs $\pm$ in the normal forms of quadratic forms over $\mathbb{C}$.

`Theorem.` ***Every conic in $\mathbb{C}^n$ is equivalent to either the cylinder over a conic in $\mathbb{C}^{n-1}$, or to one of the three conics:***

$$z_1^2 + \cdots + z_n^2 = 1, \;\; z_1^2 + \cdots + z_n^2 = 0, \;\; z_n = z_1^2 + \cdots + z_{n-1}^2.$$

`Example.` Let Q be a non-degenerate quadratic form with *real* coefficients in 3 variables. According to the previous (real) classification theorem, the conic $Q(x_1, x_2, x_3) = 1$ can be transformed by a real change of coordinates into one of the 4 normal forms shown on Figure 38. The same real change of coordinates identifies the set of *complex* solutions to the equation $Q(z_1, z_2, z_3) = 1$ with that of the normal form: $\pm z_1^2 \pm z_2^2 + \pm z_3^2 = 1$. However, $-z^2$ becomes z^2 after the change $z \mapsto \sqrt{-1}z$, which identifies the set of complex solutions with the **complex sphere** in $\mathbb{C}^3$, given by the equation $z_1^2 + z_2^2 + z_3^2 = 1$. Thus, various complex conics equivalent to the complex sphere and given by equations with real coefficients, "expose" themselves in $\mathbb{R}^3$ by various *real forms*: real spheres or ellipsoids, hyperboloids of one or two sheets (as shown on Figure 38), or even remain invisible (when the set of real points is empty).

`Remark.` The same holds true in general: various hyperboloids (as well as cones or paraboloids) of the real classification theorem are real forms of complex conics defined by the same equations. They become equivalent when complex changes of coordinates are allowed. In this sense, the three normal forms of the last theorem represent hyperboloids, cones and paraboloids of the previous one.

EXERCISES

307. Find the place of surfaces $x_1x_2+x_2x_3=\pm 1$ and $x_1x_2+x_2x_3+x_3x_1=\pm 1$ in the classification of conics in $\mathbb{R}^3$.

308. Examine normal forms of hyperboloids in $\mathbb{R}^4$ and find out how many connected components ("sheets") each of them has. ✓

309. Find explicitly a $\mathbb{C}$-linear transformation that identifies the sets of complex solutions to the equations $uv=1$ and $x^2+y^2=1$.

310. Find the rank of the quadratic form $z_1^2+2iz_1z_2-z_2^2$.

311. Classify conics in $\mathbb{C}^2$ up to linear inhomogeneous transformations. ✓

312. Find the place of the complex conic $z_1^2-2iz_1z_2-z_2^2=iz_1+z_2$ in the classification of conics in $\mathbb{C}^2$. ✓

313. Classify all conics in $\mathbb{C}^3$ up to linear inhomogeneous transformations.

314. Prove that there are $3n-1$ equivalence classes of conics in $\mathbb{C}^n$.

Hermitian and anti-Hermitian forms

In Chapter 2, at the end of section *"Matrices"*, we established one-to-one correspondences between Hermitian, anti-Hermitian, Hermitian quadratic and anti-Hermitian quadratic forms on a *complex* vector space $\mathcal{V}$.

To recall, a **sesquilinear form** is a function $\mathcal{V}\times\mathcal{V}\to\mathbb{C}$, which is $\mathbb{C}$-linear in the 2nd argument, and anti-linear in the 1st. Such a form H is called **Hermitian-symmetric** if $H(\mathbf{w},\mathbf{z})=\overline{H(\mathbf{z},\mathbf{w})}$ for all $\mathbf{z},\mathbf{w}\in\mathcal{V}$. The corresponding **Hermitian quadratic form** is $H(\mathbf{z}):=H(\mathbf{z},\mathbf{z})$. (It is denoted by the same letter, but takes in one vector argument, and assumes real values.) In a coordinate system $\mathbf{z}=z_1\mathbf{e}_1+\cdots+z_n\mathbf{e}_n$ on $\mathcal{V}$, an Hermitian quadratic form is given by the formula

$$H(\mathbf{z})=\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{z_i}\,h_{ij}\,z_j,$$

where the coefficient matrix $H=[h_{ij}]$ is Hermitian-symmetric: $H^\dagger=H$, i.e. $\bar{h}_{ij}=h_{ji}$. The corresponding Hermitian-symmetric form has the coordinate expression

$$H(\mathbf{z}, \mathbf{w}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{z}_i \, h_{ij} w_j.$$

An **anti-Hermitian form**, by definition, is a sesquilinear form, Q, satisfying $Q(\mathbf{w}, \mathbf{z}) = -\overline{Q(\mathbf{z}, \mathbf{w})}$ for all $\mathbf{z}, \mathbf{w} \in \mathcal{V}$. Every such form (and the corresponding **anti-Hermitian quadratic form** $Q(\mathbf{z}) := Q(\mathbf{z}, \mathbf{z})$) is obtained by multiplication by $\sqrt{-1}$ from an Hermitian-symmetric (respectively Hermitian quadratic) form, and *vice versa.*

`Theorem.` ***Every Hermitian quadratic form H in $\mathbb{C}^n$ can be transformed by a $\mathbb{C}$-linear change of coordinates to exactly one of the normal forms***

$$|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - \cdots - |z_{p+q}|^2, \quad 0 \le p + q \le n.$$

`Proof.` It is the same as in the case of the Inertia Theorem for real quadratic forms. We pick a vector $\mathbf{f}_1$ such that $H(\mathbf{f}_1) = \pm 1$, and consider the subspace $\mathcal{V}_1$ consisting of all vectors H-orthogonal to $\mathbf{f}_1$: $\mathcal{V}_1 = \{\mathbf{z} \mid H(\mathbf{f}_1, \mathbf{z}) = 0\}$. It does not contain $\mathbf{f}_1$ (since $H(\mathbf{f}_1, \mathbf{f}_1) = H(\mathbf{f}_1) \neq 0$), and has therefore complex codimension 1. We consider the Hermitian form obtained by restricting H to $\mathcal{V}_1$ and proceed the same way, i.e. pick a vector $\mathbf{f}_2 \in \mathcal{V}_1$ such that $H(\mathbf{f}_2) = \pm 1$, and pass to the subspace $\mathcal{V}_2$ consisting of all vectors of $\mathcal{V}_1$ which are H-orthogonal to $\mathbf{f}_2$. The process stops when we reach a subspace $\mathcal{V}_r$ of codimension r in $\mathbb{C}^n$ such that the restriction of the form H to $\mathcal{V}_r$ vanishes identically. Then we pick any basis $\{\mathbf{f}_{r+1}, \ldots, \mathbf{f}_n\}$ in $\mathcal{V}_r$. The vectors $\mathbf{f}_1, \ldots, \mathbf{f}_n$ form a basis in $\mathbb{C}^n$ which is H-orthogonal (since $H(\mathbf{f}_i, \mathbf{f}_j) = 0$ for all $i < j$ by construction), and $H(\mathbf{f}_i, \mathbf{f}_i) = \pm 1$ (for $i \le r$) or $= 0$ for $i > r$. Reordering the vectors $\mathbf{f}_1, \ldots, \mathbf{f}_r$ so that those with the values $+1$ come first, we obtain the required normal form for H, where $p + q = r$.

To prove that the normal forms with different pairs of values of p and q are non-equivalent to each other, we show (the same way as in the case of real quadratic forms) that ***the number p (q) of positive (respectively negative) squares in the normal form is equal to the maximal dimension of a subspace where the Hermitian form H (respectively $-H$) is positive definite.*** □

`Corollary 1.` ***An anti-Hermitian quadratic form Q in $\mathbb{C}^n$ can be transformed by a $\mathbb{C}$-linear change of coordinates to exactly one of the normal forms***

$$i|z_1|^2 + \cdots + i|z_p|^2 - i|z_{p+1}|^2 - \cdots - i|z_{p+q}|^2, \quad 0 \le p + q \le n.$$

Using matrix notation, one expresses a sesquilinear form with the coefficient matrix T by the matrix product formula $\mathbf{z}^\dagger T\mathbf{w}$, where $\mathbf{w}$ is a column, and $\mathbf{z}^\dagger$ is the row Hermitian-adjoint to the column $\mathbf{z}$, i.e. obtained from it by transposition and complex conjugation. Applying a $\mathbb{C}$-linear change of variables $\mathbf{z} = C\mathbf{z}', \mathbf{w} = C\mathbf{w}'$, we find

$$(\mathbf{z}')^\dagger T'\mathbf{w}' = \mathbf{z}^\dagger T\mathbf{w} = \mathbf{z}^\dagger C^\dagger TC\mathbf{w}, \quad \text{i.e. } T' = C^\dagger TC.$$

`Corollary 2.` ***Any Hermitian (anti-Hermitian) matrix can be transformed to exactly one of the normal forms***

$$\begin{bmatrix} I_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \left(\textbf{\textit{respectively}} \quad \begin{bmatrix} iI_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -iI_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right).$$

by transformations of the form $T \mapsto C^\dagger TC$ ***defined by invertible complex matrices*** C.

It follows that $p + q$ is equal to the rank of the coefficient matrix of the (anti-)Hermitian form.

EXERCISES

315. Check that $T^\dagger = T^t$ if and only if T is real.

316. Show that diagonal entries of an Hermitian matrix are real, and of anti-Hermitian imaginary.

317. Find all complex matrices which are symmetric and anti-Hermitian simultaneously. ✓

318.* Prove that a sesquilinear form T of $\mathbf{z}, \mathbf{w} \in \mathcal{V}$ can be expressed in terms of its values at $\mathbf{z} = \mathbf{w}$, and find such an expression. ϟ ✓

319. Define sesquilinear forms $T : \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}$ of pairs of vectors $(\mathbf{z}, \mathbf{w})$ taken from two different spaces, and prove that $T(\mathbf{z}, \mathbf{w}) = \langle \mathbf{z}, T\mathbf{w}\rangle$, where T is the $m \times n$-matrix of coefficients of the form, and $\langle \cdot, \cdot \rangle$ is the standard Hermitian dot-product in $\mathbb{C}^m$. ϟ

320. Prove that under changes of variables $\mathbf{v} = D\mathbf{v}'$, $\mathbf{w} = C\mathbf{w}'$ the coefficient matrices of sesquilinear forms are transformed as $P \mapsto D^\dagger PC$.

321. Prove that $\langle A\mathbf{z}, \mathbf{w}\rangle = \langle \mathbf{z}, B\mathbf{w}\rangle$ for all $\mathbf{z} \in \mathbb{C}^m$, $\mathbf{w} \in \mathbb{C}^n$ if and only if $A = B^\dagger$. Here $\langle \cdot, \cdot \rangle$ denote *Hermitian* dot-products in $\mathbb{C}^n$ or $\mathbb{C}^m$. ϟ

322. Prove that $(AB)^\dagger = B^\dagger A^\dagger$. ϟ

323. Prove that for (anti-)Hermitian matrices A and B, the **commutator** matrix $AB - BA$ is anti-Hermitian.

324. Find out which of the following forms are Hermitian or anti-Hermitian and transform them to the appropriate normal forms: ϟ

$$\bar{z}_1 z_2 - \bar{z}_2 z_1, \quad \bar{z}_1 z_2 + \bar{z}_2 z_1, \quad \bar{z}_1 z_1 + i\bar{z}_2 z_1 - i\bar{z}_1 z_2 + \bar{z}_2 z_2.$$

Sylvester's rule

Let H be an Hermitian $n \times n$-matrix. Denote by $\Delta_0 = 1$, $\Delta_1 = h_{11}$, $\Delta_2 = h_{11}h_{22} - h_{12}h_{21}$, ..., $\Delta_n = \det H$ the minors formed by the intersection of the first k rows and columns of H, $k = 1, 2, \dots, n$ (Figure 40). They are called **leading minors** of the matrix H. Note that $\det H = \det H^t = \det \bar{H} = \overline{\det H}$ is real, and the same is true for each Δ_k, since it is the determinant of an Hermitian $k \times k$-matrix. The following result is due to the English mathematician James **Sylvester** (1814–1897).

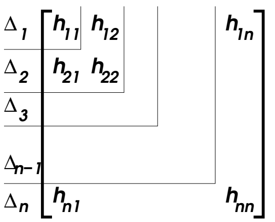

Figure 40

Theorem. ***Suppose that an Hermitian*** $n \times n$***-matrix*** H ***has non-zero leading minors. Then the negative inertia index of the corresponding Hermitian form is equal to the number of sign changes in the sequence*** $\Delta_0, \Delta_1, \dots, \Delta_n$.

Remark. The hypothesis that $\det H \neq 0$ means that the Hermitian form is **non-degenerate**, or equivalently, that its kernel is trivial. In other words, for each non-zero vector $\mathbf{x}$ there exists $\mathbf{y}$ such that $H(\mathbf{x}, \mathbf{y}) \neq 0$. Respectively, the assumption that all leading minors are non-zero means that *restrictions of the Hermitian forms to all spaces of the standard coordinate flag*

$$\text{Span}(\mathbf{e}_1) \subset \text{Span}(\mathbf{e}_1, \mathbf{e}_2) \subset \dots \subset \text{Span}(\mathbf{e}_1, \dots, \mathbf{e}_k) \subset \dots$$

are non-degenerate. The proof of the theorem consists in classifying such Hermitian forms up to linear changes of coordinates that *preserve* the flag.

Proof. As before, we inductively construct an H-orthogonal basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ and normalize the vectors so that $H(\mathbf{f}_i) = \pm 1$, requiring however that each $\mathbf{f}_k \in \text{Span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$. When such vectors $\mathbf{f}_1, \dots, \mathbf{f}_{k-1}$ are already found, the vector $\mathbf{f}_k$, H-orthogonal to them, can be found (by the Rank Theorem) in the k-dimensional space of the flag, and can be assumed to satisfy $H(\mathbf{f}_k) = \pm 1$, since the Hermitian form on this space is non-degenerate. Thus, ***an Hermitian form non-degenerate on each space of the standard coordinate flag can be transformed to one*** (and in fact exactly one) ***of the 2^n normal forms $\pm|z_1|^2 \pm \cdots \pm |z_n|^2$ by a linear change of coordinates preserving the flag.***

In matrix form, this means that there exists an invertible *upper triangular matrix* C such that $D = C^\dagger HC$ is diagonal with all diagonal entries equal to ± 1. Note that transformations of the form $H \mapsto C^\dagger HC$ may change the determinant but preserve its sign:

$$\det(C^\dagger HC) = (\det C^\dagger)(\det H)(\det C) = \det H |\det C|^2.$$

When C is upper triangular, the same holds true for all leading minors, i.e. each Δ_k has the same sign as the leading $k \times k$-minor of the diagonal matrix D with the diagonal entries $d_1, \dots, d_n$ equal ± 1. The latter minors form the sequence $1, d_1, d_1d_2, \dots, d_1 \dots d_k, \dots$, where the sign is changed each time as $d_k = -1$. Thus the total number of sign changes is equal to the number of negative squares in the normal form. $\square$

When the form H is positive definite, its restrictions to any subspace is positive definite and hence non-degenerate automatically. We obtain the following corollaries.

`Corollary 1.` ***Any positive definite Hermitian form in $\mathbb{C}^n$ can be transformed into $|z_1|^2 + \cdots + |z_n|^2$ by a linear change of coordinates preserving a given complete flag.***

`Corollary 2.` ***An Hermitian form in $\mathbb{C}^n$ is positive definite if and only if all of its leading minors are positive.***

Note that the standard basis of $\mathbb{C}^n$ is **orthonormal** with respect to the Hermitian dot product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum \bar{x}_i y_i$, i.e. $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ for $i \neq i$, and $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = 1$.

`Corollary 3.` ***Every positive definite Hermitian form in $\mathbb{C}^n$ has an orthonormal basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ such that $\mathbf{f}_k \in$ Span$(\mathbf{e}_1, \dots, \mathbf{e}_k)$.***

`Remarks.` (1) The process of replacing a given basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ with a new basis, orthonormal with respect to a given positive definite Hermitian form and such that each $\mathbf{f}_k$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_k$, is called **Gram–Schmidt orthogonalization**.

(2) Results of this subsection hold true for quadratic forms in $\mathbb{R}^n$. Namely, our reasoning can be easily adjusted to this case. Note also that every real symmetric matrix is Hermitian.

EXERCISES

325. Prove that for every symmetric matrix Q all of whose leading minors are non-zero there exists a *unipotent* upper triangular matrix C such that $D = C^t QC$ is diagonal, and express the diagonal entries of D in terms of the leading minors. ✓

326. Use Sylvester's rule to find inertia indices of quadratic forms: ✓

$$x_1^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_4, \quad x_1x_2 - x_2^2 + x_3^2 + 2x_2x_4 + x_4^2.$$

327. Compute determinants and inertia indices of quadratic forms:

$$x_1^2 - x_1x_2 + x_2^2, \quad x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3.$$

328. Prove positivity of the quadratic form $\sum_{i=1}^n x_i^2 - \sum_{1\le i<n} x_i x_{i+1}$.

329.* Prove that when the square of a linear form is added to a positive quadratic form, the determinant of the coefficient matrix increases. ϟ

330.* Prove that a non-degenerate anti-symmetric bilinear form $A(\mathbf{x}, \mathbf{y})$ in $\mathbb{K}^{2n}$ is equivalent to

$$(x_1y_2 - x_2y_1) + \cdots + (x_{2n-1}y_{2n} - x_{2n}y_{2n-1}),$$

i.e. to $x_1 \wedge x_2 + \cdots + x_{2n-1} \wedge x_{2n}$ in the exterior form notation. Namely, pick two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $A(\mathbf{e}_1, \mathbf{e}_2) = 1$, show that A is non-degenerate on the subspace $\{\mathbf{x} \in \mathbb{R}^{2n} \mid A(\mathbf{e}_1, \mathbf{x}) = A(\mathbf{e}_2, \mathbf{x}) = 0\}$, and continue by induction.

331. Derive from the previous exercise that the determinant of the coefficient matrix of an anti-symmetric bilinear form is equal to the square of its Pfaffian.

2 Euclidean Geometry

Euclidean spaces

Let $\mathcal{V}$ be a real vector space. A **Euclidean inner product** (or **Euclidean structure**) on $\mathcal{V}$ is defined as a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. A real vector space equipped with a Euclidean inner product is called a **Euclidean space**. A Euclidean inner product allows one to talk about distances between points and angles between directions:

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle}, \quad \cos\theta(\mathbf{x}, \mathbf{y}) := \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{|\mathbf{x}|\ |\mathbf{y}|}.$$

It follows from the Inertia Theorem that ***every finite dimensional Euclidean vector space has an orthonormal basis***. In coordinates corresponding to an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ the inner product is given by the standard formula:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i,j=1}^{n} x_i y_j \langle \mathbf{e}_i, \mathbf{e}_j \rangle = x_1 y_1 + \cdots + x_n y_n.$$

Thus, every Euclidean space $\mathcal{V}$ of dimension n can be identified with the **coordinate Euclidean space** $\mathbb{R}^n$ by an isomorphism $\mathbb{R}^n \to \mathcal{V}$ respecting inner products. Such an isomorphism is not unique, but can be composed with any invertible linear transformation $U : \mathcal{V} \to \mathcal{V}$ preserving the Euclidean structure:

$$\langle U\mathbf{x}, U\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathcal{V}.$$

Such transformations are called **orthogonal**.

A Euclidean structure on a vector space $\mathcal{V}$ allows one to identify the space with its dual $\mathcal{V}^*$ by the rule that to a vector $\mathbf{v} \in \mathcal{V}$ assigns the linear function on $\mathcal{V}$ whose value at a point $\mathbf{x} \in \mathcal{V}$ is equal to the inner product $\langle \mathbf{v}, \mathbf{x} \rangle$. Respectively, given a linear map $A : \mathcal{V} \to \mathcal{W}$ between Euclidean spaces, the adjoint map $A^t : \mathcal{W}^* \to \mathcal{V}^*$ can be considered as a map between the spaces themselves: $A^t : \mathcal{W} \to \mathcal{V}$. The defining property of the adjoint map reads:

$$\langle A^t \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{w}, A\mathbf{v} \rangle \quad \text{for all } \mathbf{v} \in \mathcal{V} \text{ and } \mathbf{w} \in \mathcal{W}.$$

Consequently matrices of adjoint maps A and A^t with respect to orthonormal bases of the Euclidean spaces $\mathcal{V}$ and $\mathcal{W}$ are transposed to each other.

As in the case of Hermitian spaces, one easily derives that a linear transformation $U : \mathcal{V} \to \mathcal{V}$ is orthogonal if and only if $U^{-1} = U^t$. In the matrix form, the relation $U^tU = I$ means that columns of U form an orthonormal set in the coordinate Euclidean space.

Our goal here is to develop the spectral theory for **real normal operators**, i.e. linear transformations $A : \mathcal{V} \to \mathcal{V}$ on a Euclidean space commuting with their transposed operators: $A^tA = AA^t$. Symmetric ($A^t = A$), anti-symmetric ($A^t = -A$), and orthogonal transformations are examples of normal operators in Euclidean geometry.

The right way to proceed is to consider Euclidean geometry as Hermitian geometry, equipped with an additional, *real* structure, and apply the Spectral Theorem of Hermitian geometry to real normal operators extended to the complex space.

EXERCISES

382. Prove the Cauchy-Schwartz inequality for Euclidean inner products: $\langle \mathbf{x}, \mathbf{y}\rangle^2 \leq \langle \mathbf{x}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{y}\rangle$, strictly, unless $\mathbf{x}$ and $\mathbf{y}$ are proportional, and derive from this that the angle between non-zero vectors is well-defined. ↯

383. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, put $\langle \mathbf{x}, \mathbf{x}\rangle = \sum_{i=1}^n 2x_i^2 - 2\sum_{i=1}^{n-1} x_i x_{i+1}$. Show that the corresponding symmetric bilinear form defines on $\mathbb{R}^n$ a Euclidean structure, and find the angles between the standard coordinate axes in $\mathbb{R}^n$. ✓

384. Prove that $\langle \mathbf{x}, \mathbf{x}\rangle := 2\sum_{i\leq j} x_i x_j$ defines in $\mathbb{R}^n$ a Euclidean structure, find pairwise angles between the standard coordinate axes, and show that permutations of coordinates define orthogonal transformations. ↯

385. In the standard Euclidean space $\mathbb{R}^{n+1}$ with coordinates $x_0, \dots, x_n$, consider the hyperplane H given by the equation $x_0 + \cdots + x_n = 0$. Find explicitly a basis $\{\mathbf{f}_i\}$ in H, in which the Euclidean structure has the same form as in the previous exercise, and then yet another basis $\{\mathbf{h}_i\}$ in which it has the same form as in the exercise preceding it. ✓

386. Prove that if U is orthogonal, then $\det U = \pm 1$. ↯

387. Provide a geometric description of orthogonal transformations of the Euclidean plane. Which of them have determinant 1, and which -1? ✓

388. Prove that an $n \times n$-matrix U defines an orthogonal transformation in the standard Euclidean space $\mathbb{R}^n$ if and only if the columns of U form an orthonormal basis.

389. Show that rows of an orthogonal matrix form an orthonormal basis.

Complexification

Since $\mathbb{R} \subset \mathbb{C}$, every complex vector space can be considered as a real vector space simply by "forgetting" that one can multiply by non-real scalars. This operation is called **realification**; applied to a $\mathbb{C}$-vector space $\mathcal{V}$, it produces an $\mathbb{R}$-vector space, denoted $\mathcal{V}^{\mathbb{R}}$, of real dimension twice the complex dimension of $\mathcal{V}$.

In the reverse direction, to a real vector space $\mathcal{V}$ one can associate a complex vector space, $\mathcal{V}^{\mathbb{C}}$, called the **complexification** of $\mathcal{V}$. As a real vector space, it is the direct sum of two copies of $\mathcal{V}$:

$$\mathcal{V}^{\mathbb{C}} := \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{V}\}.$$

Thus, the addition is performed componentwise, while the multiplication by complex scalars $\alpha + i\beta$ is introduced with the thought in mind that $(\mathbf{x}, \mathbf{y})$ stands for $\mathbf{x} + i\mathbf{y}$:

$$(\alpha + i\beta)(\mathbf{x}, \mathbf{y}) := (\alpha\mathbf{x} - \beta\mathbf{y}, \beta\mathbf{x} + \alpha\mathbf{y}).$$

This results in a $\mathbb{C}$-vector space $\mathcal{V}^{\mathbb{C}}$ whose complex dimension equals the real dimension of $\mathcal{V}$. Note that $i(\mathbf{y}, 0) = (0, \mathbf{y})$. Therefore $(\mathbf{x}, \mathbf{y})$ can really be written as $\mathbf{x} + i\mathbf{y}$ is one assumes that $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.

The complex space $\mathcal{V}^{\mathbb{C}}$ carries the operation of **complex conjugation** $\mathbf{z} = \mathbf{x} + i\mathbf{y} \mapsto \bar{\mathbf{z}} := \mathbf{x} - i\mathbf{y}$. It is $\mathbb{R}$-linear (i.e. is a linear transformation on the realification of the complexification), but is not $\mathbb{C}$-linear. Namely, $\overline{\mathbf{z} + \mathbf{w}} = \bar{\mathbf{z}} + \bar{\mathbf{w}}$, but $\overline{\lambda\mathbf{z}} = \bar{\lambda}\bar{\mathbf{z}}$, i.e. it is anti-linear (or half-linear) relative to the multiplication by scalars $\lambda \in \mathbb{C}$.

A real linear transformation $A : \mathcal{V} \to \mathcal{V}$ extends canonically to a complex linear transformation $A^{\mathbb{C}} : \mathcal{V}^{\mathbb{C}} \to \mathcal{V}^{\mathbb{C}}$ which we will, abusing notation, often denote by A again, because it is defined as $A^{\mathbb{C}}(\mathbf{x}, \mathbf{y}) := (A\mathbf{x}, A\mathbf{y})$. Simply speaking, a real matrix can be considered as a complex one.

By the same token, a real quadratic (bilinear, symmetric, anti-symmetric) form on $\mathcal{V}$ is canonically extended to a complex quadratic ($\mathbb{C}$-bilinear, symmetric, anti-symmetric) form on $\mathcal{V}^{\mathbb{C}}$. What we need, however, is the extension of a Euclidean inner product on $\mathcal{V}$ (not to a $\mathbb{C}$-bilinear, but) to an Hermitian inner product on $\mathcal{V}^{\mathbb{C}}$. Namely, for all $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{V}$, put

$$\langle \mathbf{x} + i\mathbf{y}, \mathbf{x}' + i\mathbf{y}' \rangle := \langle \mathbf{x}, \mathbf{x}' \rangle + \langle \mathbf{y}, \mathbf{y}' \rangle + i\langle \mathbf{x}, \mathbf{y}' \rangle - i\langle \mathbf{y}, \mathbf{x}' \rangle.$$

It is straightforward to check that this form on $\mathcal{V}^{\mathbb{C}}$ is sesquilinear and Hermitian-symmetric. When $\langle \cdot, \cdot \rangle$ is positive definite in $\mathcal{V}$, so is its extension to $\mathcal{V}^{\mathbb{C}}$, because $\langle \mathbf{x} + i\mathbf{y}, \mathbf{x} + i\mathbf{y} \rangle = |\mathbf{x}|^2 + |\mathbf{y}|^2$.

The same way as the Hermitian-transposed of a square matrix coincides with its ordinary transposed when the matrix is real, the Hermitian-adjoint $A^\dagger$ of (the complexification of) a real operator $A : \mathcal{V} \to \mathcal{V}$ coincides with (the complexification of) A^t. That is, if $\langle A^t \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, A\mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, then $\langle A^t \mathbf{z}, \mathbf{w}\rangle = \langle \mathbf{z}, A\mathbf{w}\rangle$ for all $\mathbf{z}, \mathbf{w} \in \mathcal{V}^{\mathbb{C}}$. In particular, complexifications of **orthogonal** $(U^{-1} = U^t)$, **symmetric** $(A^t = A)$, **anti-symmetric** $(A^t = -A)$, **normal** $(A^t A = AA^t)$ operators in a Euclidean space are respectively unitary, Hermitian, anti-Hermitian, normal operators on the complexified space (with the additional property that they commute with the complex conjugation: $\overline{A\mathbf{z}} = A\bar{\mathbf{z}}$).

`Remark.` A productive (though somewhat abstract) point of view on the complexification is that it is a complex vector space, $\mathcal{W}$, equipped with an additional structure which "remembers" that the space came from a real one. The structure, called **complex conjugation**, is required to be a $\mathbb{C}$-anti-linear **involution** (the latter means that repeating it twice yields the identity). Then the real objects are those complex objects in $\mathcal{W}$ which respect this additional structure. The realification of $\mathcal{W}$ splits into the direct sum of two real subspaces, $\mathcal{V} = \{\mathbf{z} \in \mathcal{W} \mid \bar{\mathbf{z}} = \mathbf{z}\}$ and $i\mathcal{V} = \{\mathbf{z} \in \mathcal{W} \mid \bar{\mathbf{z}} = -\mathbf{z}\}$. Thus, $\mathcal{W}$ is identified with $\mathcal{V}^{\mathbb{C}}$, where $\mathcal{V}$ consists of all point fixed by the complex conjugation. A complex linear transformation $A : \mathcal{W} \to \mathcal{W}$ is real whenever it commutes with the complex conjugation. For this, A must preserve the real subspace $\mathcal{V} \subset \mathcal{W}$, and A coincides with the complexification of its restriction to $\mathcal{V}$.

`Example.` Take $\mathcal{W} = \mathbb{C}^n$, with the operation of complex conjugation acting componentwise, i.e. for $\mathbf{z} \in \mathbb{C}^n$ with components $z_\alpha = x_\alpha + iy_\alpha$, the components of $\bar{\mathbf{z}}$ are $x_\alpha - iy_\alpha$. Then the real subspace $\mathcal{V} = \mathbb{R}^n = \{\mathbf{z} \in \mathbb{C}^n \mid y_\alpha = 0 \text{ for all } \alpha\}$, and $\mathbb{C}^n = (\mathbb{R}^n)^{\mathbb{C}}$. Complexifying the standard dot-product $\langle \mathbf{x}, \mathbf{y}\rangle = x_1 y_1 + \cdots + x_n y_n$ in $\mathbb{R}^n$, we end up with the standard Hermitian dot-product in $\mathbb{C}^n$: $\langle \mathbf{z}, \mathbf{w}\rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n$. Note that complex $n \times n$-matrices have the form $A + iB$ where A, B are arbitrary real $n \times n$-matrices. Those which commute with the operation of complex conjugation on $\mathbb{C}^n$ must have $B = 0$ (because multiplication by i anti-commutes with it), i.e. must be real (in every sense of the word).

EXERCISES

390. Consider $\mathbb{C}^n$ as a real vector space, and describe its complexification.

391. Check that complex conjugation in $\mathcal{V}^{\mathbb{C}}$ anti-commutes with the multiplication by i.

392. Check that the formula for the canonical extension of a bilinear form $\langle \cdot, \cdot \rangle$ from a real vector $\mathcal{V}$ to its complexification yields a sesquilinear form indeed. How would the formula change for the extension to be $\mathbb{C}$-bilinear?

393. Verify that $\langle A^t \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{z}, A\mathbf{w} \rangle$ for all $\mathbf{z}, \mathbf{w} \in \mathcal{V}^{\mathbb{C}}$, where A^t is the real adjoint to A on $\mathcal{V}$, and $\langle \cdot, \cdot \rangle$ is the canonical extension to $\mathcal{V}^{\mathbb{C}}$ of a Euclidean inner product on $\mathcal{V}$.

394. On the complex line $\mathbb{C}^1$, find all involutions anti-commuting with the multiplication by i.

395. Show that a linear involution on a (real or complex) vector space has two complementary eigenspaces corresponding to the eigenvalues ± 1. Show that when the involution is $\mathbb{R}$-linear but acts on a complex vector space and anti-commutes with the multiplication by i, then the (real!) eigenspaces have the same dimension, and the multiplication by i interchanges them.

The Real Spectral Theorem

`Theorem.` ***Let $\mathcal{V}$ be a Euclidean space, and $A : \mathcal{V} \to \mathcal{V}$ a normal operator. Then in the complexification $\mathcal{V}^{\mathbb{C}}$, there exists an orthonormal basis of eigenvectors of $A^{\mathbb{C}}$ which is invariant under complex conjugation and such that the eigenvalues corresponding to conjugated eigenvectors are conjugated.***

`Proof.` Applying the complex Spectral Theorem to the normal operator $B = A^{\mathbb{C}}$, we obtain a decomposition of the complexified space $\mathcal{V}^{\mathbb{C}}$ into a direct orthogonal sum of eigenspaces $\mathcal{W}_1, \dots, \mathcal{W}_r$ of B corresponding to distinct complex eigenvalues $\lambda_1, \dots, \lambda_r$. Note that if $\mathbf{v}$ is an eigenvector of B with an eigenvalue μ, then, since B is real, $B\bar{\mathbf{v}} = \bar{B}\bar{\mathbf{v}} = \overline{B\mathbf{v}} = \overline{\mu\mathbf{v}} = \bar{\mu}\bar{\mathbf{v}}$, i.e. $\bar{\mathbf{v}}$ is an eigenvector of B with the conjugate eigenvalue $\bar{\mu}$. This shows that if λ_i is a non-real eigenvalue, then its conjugate $\bar{\lambda}_i$ is also one of the eigenvalues of B (say, λ_j), and the corresponding eigenspaces are conjugated: $\overline{\mathcal{W}_i} = \mathcal{W}_j$. By the same token, if λ_k is real, then $\overline{\mathcal{W}_k} = \mathcal{W}_k$. The last equality means that $\mathcal{W}_k$ itself is the complexification of the real space $\{\mathbf{z} \in \mathcal{W}_k \mid \bar{\mathbf{z}} = \mathbf{z}\}$. It coincides with the space $\operatorname{Ker}(\lambda_k I - A) \subset \mathcal{V}$ of real eigenvectors of A with the eigenvalue λ_k. Thus, to construct a required orthonormal basis, we take: for each real eigenspace $\mathcal{W}_k$, a Euclidean orthonormal basis in the corresponding real eigenspace, and for each pair $\mathcal{W}_i, \mathcal{W}_j$ of complex conjugate eigenspaces, an Hermitian orthonormal basis $\{\mathbf{f}_\alpha\}$ in $\mathcal{W}_i$ and the conjugate basis $\{\overline{\mathbf{f}_\alpha}\}$ in $\mathcal{W}_j = \overline{\mathcal{W}_i}$. The vectors of all these bases altogether form an orthonormal basis of $\mathcal{V}^{\mathbb{C}}$ satisfying our requirements. $\square$

`Example 1.` Identify $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ in the usual way, and consider the operator $(x+iy)\mapsto(\alpha+i\beta)(x+iy)$ of multiplication by given complex number $\alpha+i\beta$. In the basis $1, i$, it has the matrix

$$A=\left[\begin{array}{cc}\alpha & -\beta\\ \beta & \alpha\end{array}\right].$$

Since A^t represents multiplication by $\alpha-i\beta$, it commutes with A. Therefore A is normal. It is straightforward to check that

$$\mathbf{z}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}1\\ -i\end{array}\right] \quad\text{and}\quad \bar{\mathbf{z}}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}1\\ i\end{array}\right]$$

are complex eigenvectors of A with the eigenvalues $\alpha+i\beta$ and $\alpha-i\beta$ respectively, and form an Hermitian orthonormal basis in $(\mathbb{R}^2)^{\mathbb{C}}$.

`Example 2.` If A is a linear transformation in $\mathbb{R}^n$, and λ_0 is a non-real root of its characteristic polynomial $\det(\lambda I-A)$, then the system of linear equations $A\mathbf{z}=\lambda_0\mathbf{z}$ has non-trivial solutions, which cannot be real though. Let $\mathbf{z}=\mathbf{u}+i\mathbf{v}$ be a complex eigenvector of A with the eigenvalue $\lambda_0=\alpha-i\beta$. Then $\bar{\mathbf{z}}=\mathbf{u}-i\mathbf{v}$ is an eigenvector of A with the eigenvalue $\overline{\lambda_0}=\alpha+i\beta$. Since $\lambda_0\neq\overline{\lambda_0}$, the vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ are linearly independent over $\mathbb{C}$, and hence the real vectors $\mathbf{u}$ and $\mathbf{v}$ must be linearly independent over $\mathbb{R}$. Consider the plane $\operatorname{Span}(\mathbf{u},\mathbf{v})\subset\mathbb{R}^n$. Since

$$A(\mathbf{u}+i\mathbf{v})=(\alpha-i\beta)(\mathbf{u}+i\mathbf{v})=(\alpha\mathbf{u}+\beta\mathbf{v})+i(-\beta\mathbf{u}+\alpha\mathbf{v}),$$

we conclude that A preserves this plane, and in the basis $\mathbf{u},\mathbf{v}$ in it acts by the matrix from Example 1. If we assume in addition that A is normal (with respect to the standard Euclidean structure in $\mathbb{R}^n$), then the eigenvectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ must be Hermitian orthogonal, i.e.

$$\langle\mathbf{u}-i\mathbf{v},\mathbf{u}+i\mathbf{v}\rangle=\langle\mathbf{u},\mathbf{u}\rangle-\langle\mathbf{v},\mathbf{v}\rangle+2i\langle\mathbf{u},\mathbf{v}\rangle=0.$$

We conclude that $\langle\mathbf{u},\mathbf{v}\rangle=0$ and $|\mathbf{u}|^2-|\mathbf{v}|^2=0$, i.e. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and have the same length. Normalizing the length to 1, we obtain an orthonormal basis of the A-invariant plane, in which the transformation A acts as in Example 1. The geometry of this transformation is known to us from studying the geometry of complex numbers: It is the composition of the rotation through the angle $\mathbf{arg}(\lambda_0)$ with the expansion by the factor $|\lambda_0|$. We will call such a transformation of the Euclidean plane a **complex multiplication** or **multiplication by a complex scalar**, λ_0.

Corollary 1. ***Given a normal operator on a Euclidean space, the space can be represented as a direct orthogonal sum of invariant lines and planes, on each of which the transformation acts as multiplication by a real or complex scalar respectively.***

Corollary 2. ***A transformation in a Euclidean space is orthogonal if and only if the space can be represented as the direct orthogonal sum of invariant lines and planes on each of which the transformation acts as multiplication by*** ± 1 ***and rotation respectively.***

Corollary 3. ***In a Euclidean space, every symmetric operator has an orthonormal basis of eigenvectors.***

Corollary 4. ***Every quadratic form in a Euclidean space of dimension*** n ***can be transformed by an orthogonal change of coordinates to exactly one of the normal forms:***

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2, \quad \lambda_1 \geq \cdots \geq \lambda_n.$$

Corollary 5. ***In a Euclidean space of dimension*** n***, every anti-symmetric bilinear form can be transformed by an orthogonal change of coordinates to exactly one of the normal forms***

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{r} \omega_i (x_{2i-1} y_{2i} - x_{2i} y_{2i-1}), \quad \omega_1 \geq \cdots \geq \omega_r > 0, \quad 2r \leq n.$$

Corollary 6. ***Every real normal matrix*** A ***can be written in the form*** $A = U^t M U$ ***where*** U ***is an orthogonal matrix, and*** M ***is block-diagonal matrix with each block either of size*** 1***, or of size*** 2 ***of the form*** $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$***, where*** $\beta > 0$.

If A ***is symmetric, then only blocks of size*** 1 ***are present (i.e.*** M ***is diagonal).***

If A ***is anti-symmetric, then blocks of size*** 1 ***are zero, and of size*** 2 ***are of the form*** $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$***, where*** $\omega > 0$.

If A ***is orthogonal, then all blocks of size*** 1 ***are equal to*** ± 1***, and blocks of size*** 2 ***have the form*** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$***, where*** $0 < \theta < \pi$.

EXERCISES

396. Prove that an operator on a Euclidean vector space is normal if and only if it is the sum of commuting symmetric and anti-symmetric operators.

397. Prove that in the complexification of a Euclidean plane, all rotations have a common basis of eigenvectors, and find these eigenvectors. ↯

398. Prove that an orthogonal transformation in $\mathbb{R}^3$ is either the rotation through an angle $\theta, 0 \leq \theta \leq \pi$, about some axis, or the composition of such a rotation with the reflection about the plane perpendicular to the axis.

399. Find an orthonormal basis in $\mathbb{C}^n$ in which the transformation defined by the cyclic permutation of coordinates: $(z_1, z_2, \dots, z_n) \mapsto (z_2, \dots, z_n, z_1)$ is diagonal, and determine the eigenvalues.

400. In the coordinate Euclidean space $\mathbb{R}^n$ with $n \leq 4$, find real and complex normal forms of orthogonal transformations defined by various permutations of coordinates.

401. Transform to normal forms by orthogonal transformations: ✓

$$\text{(a) } x_1x_2 + x_3x_4, \quad \text{(b) } 2x_1^2 - 4x_1x_2 + x_2^2 - 4x_2x_3,$$
$$\text{(c) } 3x_1^2 + 3x_2^2 + 3x_3^2 + 3x_4^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_4 + 4x_3x_4.$$

402. In Euclidean spaces, classify all operators which are both orthogonal and anti-symmetric.

403. Recall that a bilinear form on $\mathcal{V}$ is called **non-degenerate** if the corresponding linear map $\mathcal{V} \to \mathcal{V}^*$ is an isomorphism, and **degenerate** otherwise. Prove that all non-degenerate anti-symmetric bilinear forms on $\mathbb{R}^{2n}$ are equivalent to each other, and that all anti-symmetric bilinear forms on $\mathbb{R}^{2n+1}$ are degenerate.

404. Derive Corollaries 1 – 6 from the Real Spectral Theorem.

405. Let $\mathcal{U}$ and $\mathcal{V}$ be two subspaces of dimension 2 in a Euclidean 4-space. Consider the map $T : \mathcal{V} \to \mathcal{V}$ defined as the composition: $\mathcal{V} \subset \mathbb{R}^4 \to \mathcal{U} \subset \mathbb{R}^4 \to \mathcal{V}$, where the arrows are the orthogonal projections to $\mathcal{U}$ and $\mathcal{V}$ respectively. Prove that T is positive, and that its eigenvalues have the form $\mathbf{cos}\,\phi$, $\mathbf{cos}\,\psi$ where ϕ, ψ are certain angles, $0 \leq \phi, \psi \leq \pi/2$.

406. Solve **Gelfand's problem**[6]: In a Euclidean 4-space, classify pairs of planes passing through the origin up to orthogonal transformations of the space. ↯

[6] After I. M. **Gelfand** (1913–2009)

Courant's minimax principle

One of the consequences (equivalent to Corollary 4) of the Real Spectral Theorem is that a pair (Q, S) of quadratic forms in $\mathbb{R}^n$, of which the first one is positive definite, can be transformed by a linear change of coordinates to the normal form:

$$Q = x_1^2 + \cdots + x_n^2, \quad S = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2, \quad \lambda_1 \geq \cdots \geq \lambda_n.$$

The eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ form the **spectrum** of the pair (Q, S). The following result (due to Richard **Courant** (1888–1972), see [2]) gives a coordinate-free, geometric description of the spectrum (and thus implies the **Orthogonal Diagonalization Theorem** as it was stated in the Introduction).

`Theorem.` ***The k-th greatest spectral number is given by***

$$\lambda_k = \max_{\mathcal{W}:\ \dim \mathcal{W}=k}\ \min_{\mathbf{x}\in\mathcal{W}-\mathbf{0}} \frac{S(\mathbf{x})}{Q(\mathbf{x})},$$

where the maximum is taken over all k-dimensional subspaces $\mathcal{W} \subset \mathbb{R}^n$, and the minimum over all non-zero vectors in the subspace.

`Proof.` When $\mathcal{W}$ is given by the equations $x_{k+1} = \cdots = x_n = 0$, the minimal ratio $S(\mathbf{x})/Q(\mathbf{x})$ (achieved on vectors proportional to $\mathbf{e}_k$) is equal to λ_k because

$$\lambda_1 x_1^2 + \cdots + \lambda_k x_k^2 \geq \lambda_k (x_1^2 + \cdots + x_k^2) \text{ when } \lambda_1 \geq \cdots \geq \lambda_k.$$

Therefore it suffices to prove for every other k-dimensional subspace $\mathcal{W}$ the minimal ratio cannot be greater than λ_k. For this, denote by $\mathcal{V}$ the subspace of dimension $n-k+1$ given by the equations $x_1 = \cdots = x_{k-1} = 0$. Since $\lambda_k \geq \cdots \geq \lambda_n$, we have:

$$\lambda_k x_k^2 + \cdots + \lambda_n x_n^2 \leq \lambda_k (x_k^2 + \cdots + x_n^2),$$

i.e. for all non-zero vectors $\mathbf{x}$ in $\mathcal{V}$ the ratio $S(\mathbf{x})/Q(\mathbf{x}) \leq \lambda_k$. Now we invoke the dimension counting argument: $\dim \mathcal{W} + \dim \mathcal{V} = k + (n-k+1) = n+1 > \dim \mathbb{R}^n$, and conclude that $\mathcal{W}$ has a non-trivial intersection with $\mathcal{V}$. Let $\mathbf{x}$ be a non-zero vector in $\mathcal{W} \cap \mathcal{V}$. Then $S(\mathbf{x})/Q(\mathbf{x}) \leq \lambda_k$, and hence the minimum of the ratio S/Q on $\mathcal{W} - \mathbf{0}$ cannot exceed λ_k. □

Applying Theorem to the pair $(Q, -S)$ we obtain yet another characterization of the spectrum:

$$\lambda_k = \min_{\mathcal{W}:\ \dim \mathcal{W}=n-k+1}\ \max_{\mathbf{x}\in\mathcal{W}-\mathbf{0}} \frac{S(\mathbf{x})}{Q(\mathbf{x})}.$$

Formulating some applications, we assume that the space $\mathbb{R}^n$ is Euclidean, and refer to the spectrum of the pair (Q, S) where $Q = |\mathbf{x}|^2$, simply as the spectrum of S.

`Corollary 1.` ***When a quadratic form increases, its spectral numbers do not decrease: If*** $S \leq S'$ ***then*** $\lambda_k \leq \lambda'_k$ ***for all*** $k = 1, \ldots, n$.

`Proof.` Indeed, since $S/Q \leq S'/Q$, the minimum of the ratio S/Q on every k-dimensional subspace $\mathcal{W}$ cannot exceed that of S'/Q, which in particular remains true for that $\mathcal{W}$ on which the maximum of S/Q equal to λ_k is achieved.

The following is known as **Cauchy's interlacing theorem**.

`Corollary 2.` ***Let*** $\lambda_1 \geq \cdots \geq \lambda_n$ ***be the spectrum of a quadratic form*** S, ***and*** $\lambda'_1 \geq \cdots \geq \lambda'_{n-1}$ ***be the spectrum of the quadratic form*** S' ***obtained by restricting*** S ***to a given hyperplane*** $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ ***passing through the origin. Then:***

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n.$$

`Proof.` The maximum over all k-dimensional subspaces $\mathcal{W}$ cannot be smaller than the maximum (of the same quantities) over subspaces lying inside the hyperplane. This proves that $\lambda_k \geq \lambda'_k$. Applying the same argument to $-S$ and subspaces of dimension $n - k - 1$, we conclude that $-\lambda_{k+1} \geq -\lambda'_k$. $\square$

An **ellipsoid** in a Euclidean space is defined as the level-1 set $E = \{\mathbf{x} \mid S(\mathbf{x}) = 1\}$ of a positive definite quadratic form, S. It follows from the Spectral Theorem that every ellipsoid can be transformed by an orthogonal transformation to **principal axes**: a normal form

$$\frac{x_1^2}{\alpha_1^2} + \cdots + \frac{x_n^2}{\alpha_n^2} = 1, \quad 0 < \alpha_1 \leq \cdots \leq \alpha_n.$$

The vectors $\mathbf{x} = \pm\alpha_k \mathbf{e}_k$ lie on the ellipsoid, and their lengths α_k are called the **semiaxes** of E. They are related to the spectral numbers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ of the quadratic form by $\alpha_k^{-1} = \sqrt{\lambda_k}$. From Corollaries 1 and 2 respectively, we obtain:

Given two concentric ellipsoids enclosing one another, the semiaxes of the inner ellipsoid do not exceed the corresponding semiaxes of the outer: If $E' \subset E$, *then* $\alpha'_k \leq \alpha_k$ *for all* $k = 1, \ldots, n$.

The semiaxes of a given ellipsoid are interlaced by the semiaxes of any section of it by a hyperplane passing through the center: If $E' = E \cap \mathbb{R}^{n-1}$, *then* $\alpha_k \leq \alpha'_k \leq \alpha_{k+1}$ *for* $k = 1, \ldots, n-1$.

EXERCISES

407. Prove that every ellipsoid in $\mathbb{R}^n$ has n pairwise perpendicular hyperplanes of bilateral symmetry.

408. Given an ellipsoid $E \subset \mathbb{R}^3$, find a plane passing through its center and intersecting E in a circle. ϟ

409. Formulate and prove counterparts of Courant's minimax principle and Cauchy's interlacing theorem for Hermitian forms.

410. Prove that semiaxes $\alpha_1 \leq \alpha_2 \leq \ldots$ of an ellipsoid in $\mathbb{R}^n$ and semiaxes $\alpha'_k \leq \alpha'_2 \leq \ldots$ of its section by a linear subspaces of *co*dimension k are related by the inequalities: $\alpha_i \leq \alpha'_i \leq \alpha_{i+k}$, $i = 1, \ldots, n-k$.

411. From the Real Spectral Theorem, derive the Orthogonal Diagonalization Theorem as it is formulated in the Introduction, i.e. for pairs of quadratic forms on $\mathbb{R}^n$, one of which is positive definite. ϟ

Small oscillations

Let us consider the system of n identical masses m positioned at the vertices of a regular n-gon, which are cyclically connected by n identical elastic springs, and can oscillate in the direction perpendicular to the plane of the n-gon.

Assuming that the amplitudes of the oscillation are small, we can describe the motion of the masses as solutions to the following system of n second-order Ordinary Differential Equations (ODE for short) expressing Newton's law of motion (mass $\times$ acceleration = force):

$$\begin{array}{lcl} m\ddot{x}_1 & = & -k(x_1 - x_n) - k(x_1 - x_2), \\ m\ddot{x}_2 & = & -k(x_2 - x_1) - k(x_2 - x_3), \\ & \cdots & \\ m\ddot{x}_{n-1} & = & -k(x_{n-1} - x_{n-2}) - k(x_{n-1} - x_n) \\ m\ddot{x}_n & = & -k(x_n - x_{n-1}) - k(x_n - x_1). \end{array} .$$

Here $x_1, \ldots, x_n$ are the displacements of the n masses in the direction perpendicular to the plane, and k characterizes the rigidity of the springs.[7]

[7] More precisely (see Figure 43, where $n = 4$), we may assume that the springs are stretched, but the masses are confined on the vertical rods and can only slide along them without friction. When a string of length L is horizontal ($\Delta x = 0$), the stretching force T is compensated by the reactions of the rods. When $\Delta x \neq 0$, the horizontal component of the stretching force is still compensated, but the vertical component contributes to the right hand side of Newton's equations. When Δx is small, the contribution equals approximately $-T(\Delta x)/L$ (so that $k = -T/L$).

In fact the above ODE system can be read off a pair of quadratic forms: the **kinetic energy**

$$K(\dot{\mathbf{x}}) = \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} + \cdots + \frac{m\dot{x}_n^2}{2},$$

and the **potential energy**

$$P(\mathbf{x}) = k\frac{(x_1 - x_2)^2}{2} + k\frac{(x_2 - x_3)^2}{2} + \cdots + k\frac{(x_n - x_1)^2}{2}.$$

Namely, for any conservative mechanical system with quadratic kinetic and potential energy functions

$$K(\dot{\mathbf{x}}) = \frac{1}{2}\langle \dot{\mathbf{x}}, M\dot{\mathbf{x}}\rangle, \;\; P(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}, Q\mathbf{x}\rangle$$

the equations of motion assume the form

$$M\ddot{\mathbf{x}} = -Q\mathbf{x}.$$

A linear change of variables $\mathbf{x} = C\mathbf{y}$ transforms the kinetic and potential energy functions to a new form with the matrices $M' = C^tMC$ and $Q' = C^tQC$. On the other hand, the same change of variables transforms the ODE system $M\ddot{\mathbf{x}} = -Q\mathbf{x}$ to $MC\ddot{\mathbf{y}} = -QC\mathbf{y}$. Multiplying by C^t we get $M'\ddot{\mathbf{y}} = -Q'\mathbf{y}$ and see that the relationship between K, P and the ODE system is preserved. The relationship is therefore *intrinsic*, i.e. independent on the choice of coordinates.

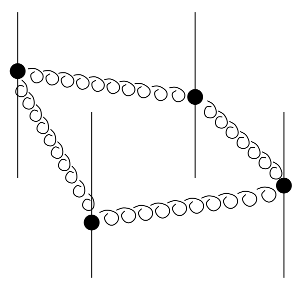

Figure 43

Since the kinetic energy is positive we can apply the Orthogonal Diagonalization Theorem in order to transform K and P simultaneously to

$$\frac{1}{2}(\dot{X}_1^2 + ... + \dot{X}_n^2), \text{ and } \frac{1}{2}(\lambda_1 X_1^2 + ... + \lambda_n X_n^2).$$

The corresponding ODE system splits into unlinked 2-nd order ODEs

$$\ddot{X}_1 = -\lambda_1 X_1, \quad ..., \quad \ddot{X}_n = -\lambda_n X_n.$$

When the potential energy is also positive, we obtain a system of n unlinked **harmonic oscillators** with frequencies $\omega = \sqrt{\lambda_1}, ..., \sqrt{\lambda_n}$.

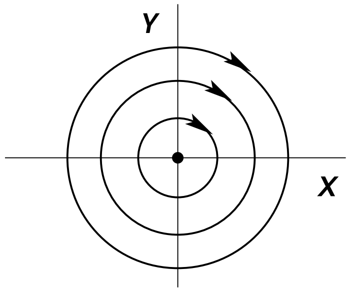

Figure 44

`Example 1:` *Harmonic oscillators.* The equation $\ddot{X} = -\omega^2 X$ has solutions

$$X(t) = A\cos\omega t + B\sin\omega t,$$

where $A = X(0)$ and $B = \dot{X}(0)/\omega$ are arbitrary real constants. It is convenient to plot the solutions on the **phase plane** with coordinates $(X, Y) = (X, \dot{X}/\omega)$. In such coordinates, the equations of motion assume the form

$$\begin{array}{rcr} \dot{X} & = & \omega Y \\ \dot{Y} & = & -\omega X \end{array},$$

and the solutions

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} X(0) \\ Y(0) \end{bmatrix}.$$

In other words (see Figure 44), the motion on the phase plane is described as ***clockwise rotation with the angular velocity*** ω. Since there is one trajectory through each point of the phase plane, the general theory of Ordinary Differential Equations (namely, the theorem about uniqueness and existence of solutions with given initial conditions) guarantees that these are all the solutions to the ODE $\ddot{X} = -\omega^2 X$.

Let us now examine the behavior of our system of n masses cyclically connected by the springs. To find the common orthogonal basis of the pair of quadratic forms K and P, we first note that, since K is proportional to the standard Euclidean structure, it suffices to find an orthogonal basis of eigenvectors of the symmetric matrix Q.

In order to give a concise description of the ODE system $m\ddot{\mathbf{x}} = Q\mathbf{x}$, introduce operator $T : \mathbb{R}^n \to \mathbb{R}^n$ which cyclically shifts the coordinates: $T(x_1, x_2, \ldots, x_n)^t = (x_2, \ldots, x_n, x_1)$. Then $Q = k(T + T^{-1} - 2I)$. Note that the operator T is obviously orthogonal, and hence unitary in the complexification $\mathbb{C}^n$ of the the space $\mathbb{R}^n$. We will now construct its basis of eigenvectors, which should be called the **Fourier basis**.[8] Namely, let $x_k = \zeta^k$ where $\zeta^n = 1$. Then the sequence $\{x_k\}$ is repeating every n terms, and $x_{k+1} = \zeta x_k$ for all $k \in \mathbb{Z}$. Thus $T\mathbf{x} = \zeta\mathbf{x}$, where $\mathbf{x} = (\zeta, \zeta^2, \ldots, \zeta^n)t$. When $\zeta = e^{2\pi\sqrt{-1}\, l/n}$, $l = 1, 2, \ldots, n$ runs various nth roots of unity, we obtain n eigenvectors of the operator T, which corresponds to different eigenvalues, and hence are linearly independent. They are automatically pairwise Hermitian orthogonal (since T is unitary), and happen to have the same Hermitian inner square, equal to n. Thus, when divided by $\sqrt{n}$, these vectors form an orthonormal basis in $\mathbb{C}^n$. Besides, this basis is invariant under complex conjugation (because replacing the eigenvalue ζ with $\bar{\zeta}$ also conjugates the corresponding eigenvector).

Now, applying this to $Q = k(T + T^{-1} - 2I)$, we conclude that Q is diagonal in the Fourier basis with the eigenvalues

$$k(\zeta + \zeta^{-1} - 2) = 2k(\mathbf{cos}(2\pi l/n) - 1) = -4k\,\mathbf{sin}^2\,\pi l/n, \quad l = 1, 2, \ldots, n.$$

When $\zeta \neq \bar{\zeta}$, this pair of roots of unity yields the same eigenvalue of Q, and the real and imaginary parts of the Fourier eigenvector $\mathbf{x} = (\zeta, \ldots, \zeta^n)^t$ span in $\mathbb{R}^n$ the 2-dimensional eigenplane of the operator Q. When $\zeta = 1$ or -1 (the latter happens only when n is even), the corresponding eigenvalue of Q is 0 and $-4k$ respectively, and the eigenspace is 1-dimensional (spanned the respective Fourier vectors $(1, \ldots, 1)^t$ and $(-1, 1, \ldots, -1, 1)^t$. The whole systems decomposes into superposition of independent "modes of oscillation" (patterns) described by the equations

$$\ddot{X}_l = -\omega_l^2 X_l, \quad \text{where } \omega_l = 2\sqrt{\frac{k}{m}}\,\mathbf{sin}\,\pi\frac{l}{n}, \quad l = 1, \ldots, n.$$

[8] After Joseph **Fourier** (1768–1830), and by analogy with the theory of Fourier series.

`Example 2:` $n = 4$. Here $\zeta = 1, -1, \pm i$. The value $\zeta = 1$ corresponds to the eigenvector $(1, 1, 1, 1)$ and the eigenvalue 0. This "mode of oscillation" is described by the ODE $\ddot{X} = 0$, and actually corresponds to the steady translation of the chain as a whole with the constant speed (Figure 45a). The value $\zeta = -1$ corresponds to the eigenvector $(-1, 1, -1, 1)$ (Figure 45b) with the frequency of oscillation $2\sqrt{k/m}$. The values $\zeta = \pm i$ correspond to the eigenvectors $(\pm i, -1, \mp i, 1)$. Their real and imaginary parts $(0, -1, 0, 1)$ and $(1, 0, -1, 0)$ (Figures 45cd) span the plane of modes of oscillation with the same frequency $\sqrt{2k/m}$. The general motion of the system is a superposition of these four patterns.

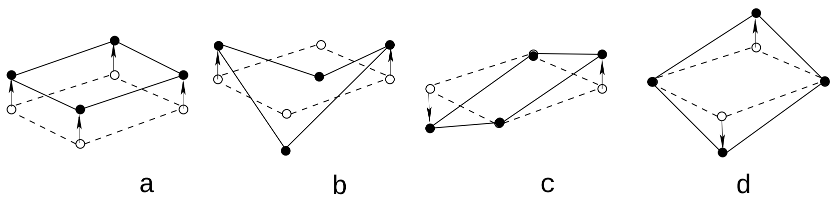

Figure 45

`Remark.` In fact the oscillatory system we've just studied can be considered as a model of sound propagation in a one-dimensional crystal. One can similarly analyze propagation of sound waves in 2-dimensional membranes of rectangular or periodic (toroidal) shape, or in similar 3-dimensional regions. Physicists often call the resulting picture — superposition of independent sinusoidal waves — an *ideal gas of phonons.* Here "ideal gas" refers to the independence of the eigen-modes of oscillation (therefore behaving as non-interacting particles of a rarefied gas), and "phonons" emphasizes that the "particles" are rather *bells* producing sound waves of various frequencies.

The mathematical aspect of this theory is even more general. The Orthogonal Diagonalization Theorem guarantees that ***any conservative mechanical system near a local minimum of potential energy is "an ideal gas of harmonic oscillators" i.e. its behavior can be described as the superposition of independent modes of harmonic oscillation.***

EXERCISES

412. A mass m is suspended on a weightless rod of length l (as a clock **pendulum**), and is swinging without friction under the action of the force of gravity mg (where g is the **gravitation constant**). Show that the Newton equation of motion of the pendulum has the form $l\ddot{x} = -g\,\mathbf{sin}\,x$, where x is the angle the rod makes with the downward vertical direction, and show that the frequency of small oscillations of the pendulum near the lower equilibrium ($x = 0$) is equal to $\sqrt{g/l}$. ϟ

413. In the mass-spring chain (studied in the text) with $n = 3$, find the frequencies and describe explicitly the modes of oscillations.

414. The same, for 6 masses positioned at the vertices of the regular hexagon (like the 6 carbon atoms in benzene molecules).

415.* Given n numbers $C_1, \dots, C_n$ (real or complex), we form from them an infinite periodic sequence $\{C_k\}$, $\dots, C_{-1}, C_0, C_1, \dots, C_n, C_{n+1}, \dots,$ where $C_{k+n} = C_k$. Let C denote the $n \times n$-matrix whose entries $c_{ij} = C_{j-i}$. Prove that all such matrices (corresponding to different n-periodic sequences) are normal, that they commute, and find their common eigenvectors. ϟ

416.* Study small oscillations of a 2-dimensional crystal lattice of toroidal shape consisting of $m \times n$ identical masses (positioned in m circular "rows" and n circular "columns", each interacting only with its four neighbors).

417. Using Courant's minimax principle, explain why a cracked bell sounds lower than the intact one.

Epilogue: Quivers

Gabriel's theorem

A figure consisting of several points connected by edges is called a **graph**. More precisely, a graph is a purely combinatorial object, which is considered given, if a finite set of its **vertices** has been specified, and the pairs of vertices connected by **edges** have been specified too. If the edges are equipped with directions, the graph is called **oriented**. A graph is called **connected** if between any two vertices there exists a path consisting of edges (regardless of their directions).

A **quiver** is a connected oriented graph. For some examples see Figure 47, which shows that we do not exclude the possibility of multiple edges connecting the same vertices, or edges connecting a vertex with itself.

Given a quiver, its **representation** consists of vector spaces assigned to the vertices, and linear maps assigned to the edges. More precisely, to each vertex v_i there should correspond a finite dimensional $\mathbb{K}$-vector space $\mathcal{V}_i$, and to each edge e_{ij} directed from v_i to v_j, there should correspond a $\mathbb{K}$-linear map $A_{ij} : \mathcal{V}_i \to \mathcal{V}_j$.

`Examples` (see Figure 47). (1) The quiver, called A_1, consists of one vertex with no edges. A representation of this quiver is simply a vector space.

(2) The quiver called $\tilde{A}_0$ consists of one vertex and one edge from this vertex to itself. A representation of this quiver is a linear map from a vector space to itself.

(3) The quiver called A_2 consists of two vertices connected by an edge. A representation of this quiver is a linear map between two vector spaces.

(4) In $\mathbb{K}^n$, consider a complete flag $\mathcal{V}^1 \subset \mathcal{V}^2 \subset \cdots \subset \mathcal{V}^{n-1} \subset \mathbb{K}^n$. It can be interpreted as a representation of the quiver consisting of n vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ (the case of $n = 5$ is shown on Figure

47 under the name A_5) connected by the edges $e_{12}, e_{23}, \ldots, e_{n-1,n}$. Indeed, the spaces of the flag correspond to the vertices, and the inclusions $\mathcal{V}^i \subset \mathcal{V}^{i+1}$ are the required linear maps. Of course, not every representation of this quiver corresponds to a flag (and in particular, the linear maps are not required to be injective).

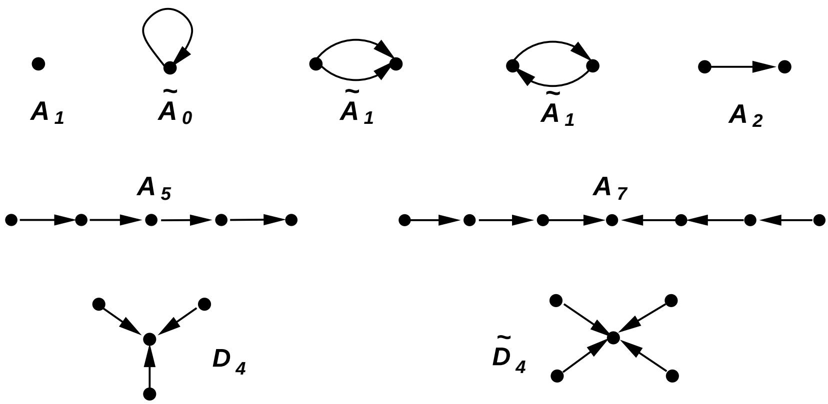

Figure 47

(5) In $\mathbb{K}^n$, consider a pair of complete flags. They can be considered as a representation

$$\mathcal{V}^1 \subset \cdots \subset \mathcal{V}^{n-1} \subset \mathbb{K}^n \supset \mathcal{W}^{n-1} \supset \cdots \supset \mathcal{W}^1$$

of the quiver, consisting of $2n-1$ vertices $v_1, \ldots, v_n, v_{n+1} \ldots v_{2n-1}$ (the case of $n = 4$ is shown on Figure 47 under the name A_7) connected by the edges $e_{12}, \ldots, e_{n-1,n}$ and $e_{2n-1,2n-2}, \ldots, e_{n+1,n}$.

(6) A triple of subspaces: $\mathcal{U}, \mathcal{V}, \mathcal{W} \subset \mathbb{K}^n$ can be considered as a representation of the quiver, denoted D_4 on Figure 47. We leave it for the reader to give examples of representations of quiver $\tilde{D}_4$, and to interpret representations of the two types of quivers $\tilde{A}_1$, shown on Figure 47.

Two representations of the same quiver are called **equivalent** if the spaces, corresponding to the vertices can be identified by isomorphisms in such a way that the corresponding maps are also identified. In greater detail, let $\mathcal{U}$ and $\mathcal{V}$ be two representations of the same quiver with vertices $\{v_i\}$. This means (see Figure 48, where the quiver is of type D_4, with a certain orientation of the edges) that we are given vector spaces $\mathcal{U}_i$ and $\mathcal{V}_i$, and for each edge e_{ij}, two linear maps: $A_{ij} : \mathcal{U}_i \to \mathcal{U}_j$, and $B_{ij} : \mathcal{V}_i \to \mathcal{V}_j$. To establish an equivalence between the representations, one needs to find *isomorphisms* $C_i : \mathcal{U}_i \to \mathcal{V}_i$ (shown on Figure 48 by vertical dashed arrows) such that the pairs of parallel horizontal and vertical arrows form **commutative squares**: $C_j A_{ij} = B_{ij} C_i$. In particular, corresponding spaces $\mathcal{U}_i, \mathcal{V}_i$ of equivalent quivers must have the same dimensions, and the corresponding maps A_{ij}, B_{ij} must have the same ranks.

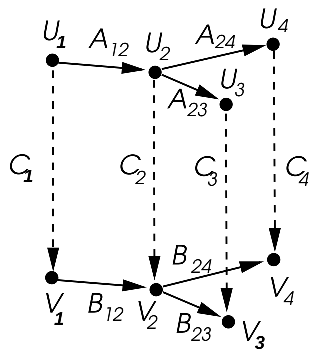

Figure 48

`Examples.` (7) Two representations of the quiver A_1 (i.e. two vector spaces) are equivalent whenever the spaces are isomorphic. As we know, this this happens exactly when the spaces have the same dimension.

(8) According to the Rank Theorem, two representations $A : \mathcal{U}_1 \to \mathcal{U}_2$ and $B : \mathcal{V}_1 \to \mathcal{V}_2$ of the same quiver $\bullet \to \bullet$ are equivalent whenever $\dim \mathcal{U}_i = \dim \mathcal{V}_i$ for $i = 1, 2$, and $\operatorname{rk} A = \operatorname{rk} B$. Indeed, when the dimensions n and m of the source and target space are fixed, and the rank $r \leq \min(m, n)$ is given, the representation is equivalent to the standard one: $E_r^{n,m} : \mathbb{K}^n \to \mathbb{K}^m$, given in coordinates by $(x_1, \dots, x_n) \to (x_1, \dots, x_r, 0, \dots, 0)$.

(9) Two representations of the quiver of type $\tilde{A}_0$ are linear maps: $A : \mathcal{U} \to \mathcal{U}$, and $B : \mathcal{V} \to \mathcal{V}$. They are equivalent, whenever there is an isomorphism $C : \mathcal{U} \to \mathcal{V}$ such that $CA = BC$, i.e. $B = CAC^{-1}$. Since spaces of the same dimension n are isomorphic to $\mathbb{K}^n$, we conclude that classification of representations of this quiver up to equivalence coincides with the classification of square matrices up to similarity transformations. When $\mathbb{K} = \mathbb{C}$, the answer is given by the Jordan Canonical Form theorem, and for $\mathbb{K} = \mathbb{R}$ by the real version of this theorem.

As illustrated by above examples, each quiver leads to a well-posed classification problem of Linear Algebra: classification of representations of the given quiver up to equivalence. Moreover, two of the four classification theorems of Linear Algebra studied in this course turn out to answer such "quiver" problems corresponding to two very special examples: quivers A_2 (the Rank Theorem), and $\tilde{A}_0$ (the Jordan Canonical Form Theorem).

One way to classify representations of a given quiver is to reduce the problem to classification of *indecomposable representations.*

Given two representations of a given quiver, $\mathcal{U} = \{\mathcal{U}_i, A_{ij}\}$ and $\mathcal{V} = \{\mathcal{V}_i, B_{ij}\}$, one can form their **direct sum** $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ by taking $\mathcal{W}_i = \mathcal{U}_i \oplus \mathcal{V}_i$ on the role of the spaces, and $C_{ij} = A_{ij} \oplus B_{ij}$ on the role of the maps. The latter means that $C_{ij} : \mathcal{U}_i \oplus \mathcal{V}_i \to \mathcal{U}_j \oplus \mathcal{V}_j$ is defined by $C_{ij}(\mathbf{u}, \mathbf{v}) = (A_{ij}\mathbf{u}, B_{ij}\mathbf{v})$. In matrix terms, if linear maps A_{ij} and B_{ij} are given by their matrices in some bases of the respective spaces $\mathcal{U}_i, \mathcal{U}_j$ and $\mathcal{V}_i, \mathcal{V}_j$, then the matrix of C_{ij} is block-diagonal:

$$C_{ij} = \begin{bmatrix} A_{ij} & 0 \\ 0 & B_{ij} \end{bmatrix}.$$

A representation which is equivalent to the direct sum of non-zero[1] representations is called **decomposable**, and **indecomposable** otherwise.

`Examples.` (10) According to the Jordan Canonical Form Theorem, every representation of quiver $\tilde{A}_0$ over $\mathbb{K} = \mathbb{C}$ is equivalent to the direct sum of Jordan cells. Each Jordan cell is in fact *indecomposable.* Indeed, a Jordan cell has a one-dimensional eigenspace, but block-diagonal matrices have at least one one-dimensional eigenspace for each of the diagonal blocks.

(11) Each representation of quiver A_2: $\bullet \to \bullet$ is equivalent to the direct sum of three indecomposable representations:

$$\mathcal{U}^{01} : \mathbb{K}^0 \to \mathbb{K}^1, \;\; \mathcal{U}^{11} : \mathbb{K}^1 \overset{\simeq}{\to} \mathbb{K}^1, \;\; \mathcal{U}^{10} : \mathbb{K}^1 \to \mathbb{K}^0.$$

This follows from the Rank Theorem. Indeed, the matrix $E_r^{n,m} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is block-diagonal, with r diagonal blocks $\mathcal{U}^{11}$ and one *zero* diagonal block of size $(m-r) \times (n-r)$. The latter, i.e. the zero map from $\mathbb{K}^{n-r}$ to $\mathbb{K}^{m-r}$ can be described as the direct sum of $n-r$ copies of $\mathcal{U}^{10}$ and $m-r$ copies of $\mathcal{U}^{01}$.

[1] The **zero** representation has all $\mathcal{U}_i = \{\mathbf{0}\}$ and consequently all maps $A_{ij} = 0$.

`Remark.` A representation $\{\mathcal{U}_i, A_{ij}\}$ of a given quiver is called **reducible** if there exists a non-trivial collection of subspaces $\mathcal{V}_i \subset \mathcal{U}_i$ which are respected by the maps A_{ij}, i.e. $A_{ij}(\mathcal{V}_i) \subset \mathcal{V}_j$. In this case, the subspaces $\mathcal{V}_i$ together with the restrictions of the maps A_{ij} to $\mathcal{V}_i$ also form a representation of the quiver: a **subrepresentation**. For instance, in a decomposable representation, each of the direct summands is a subrepresentation. A representation which does not have a non-trivial subrepresentation is called **irreducible**. For instance, $\mathcal{U}^{01}$, $\mathcal{U}^{11}$, and $\mathcal{U}^{10}$ are irreducible representations of quiver A_2. On the other hand, among Jordan cells $J_m : \mathbb{K}^m \to \mathbb{K}^m$, only the cell of size $m = 1$ is irreducible as a representation of $\tilde{A}_0$, because all Jordan cells of size $m > 1$ have non-trivial invariant subspaces. We see therefore two fundamental distinctions between the classification problems for the quivers A_2 and $\tilde{A}_0$: In the former case, there are finitely many (three) indecomposable representations, each of them is irreducible, and each representation is equivalent to a direct sum of them. In the latter case, there are infinitely many irreducible representations (they depend on parameters — eigenvalues), and indecomposable representations are not necessarily equivalent to a direct sum of irreducible ones.

`Definition.` A quiver is called **simple**, if indecomposable representations form finitely many equivalence classes.

`Example 12.` The quiver A_2 is simple, and $\tilde{A}_0$ is not.

`Theorem` (Pierre **Gabriel** [4]). ***A quiver is simple if and only if it has the form of one of the graphs*** $A_n, n \geq 1$***,*** $D_n, n \geq 4$***,*** $E_n, n = 6, 7, 8$ (see Figure 49), ***where*** n ***is the number of vertices, and the orientations of the edges are arbitrary.***

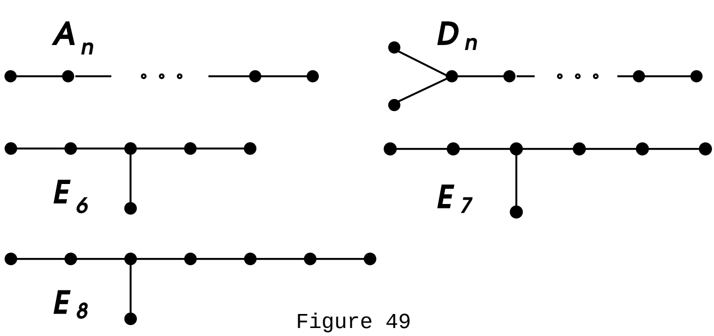

Figure 49

One statement of the theorem: that all these quivers are simple, is proved by classifying their indecomposable representations. This enthralling problem of linear algebra goes beyond our goal in this book.[2] Let us focus here on the converse statement: that only the quivers A_n, D_n, E_6, E_7, E_8 can be simple.

First, it should be clear that ***if a quiver is simple, then its representations in spaces of dimensions, not exceeding a certain bound, D, form finitely many equivalence classes.*** Indeed, let $d_1, \dots, d_n$ denote the dimensions of the vector spaces $\mathcal{U}_i$ associated to the vertices $v_1, \dots, v_n$ in a given representation $\mathcal{U}$. If this representation is equivalent to the direct sum of N indecomposable ones, then the total dimension $\sum d_i$ must be at least N. When $N > nD$, some d_i will exceed D. But if the number of types of indecomposable representations is finite, there are only finitely many ways to arrange them into direct sums of $\leq nD$ summands.

Let us now do some dimension count. Consider the problem of classification of representations of a given quiver, assuming that the dimensions $d_1, \dots, d_n$ have been fixed. In other words, the space $\mathcal{U}_i$ associated to the vertex v_i can be identified with the coordinate space $\mathbb{K}^{d_i}$ by a choice of basis. The operator $A_{ij} : \mathcal{U}_i \to \mathcal{U}_j$ corresponding to an edge e_{ij} is then described (in the chosen bases of the spaces $\mathcal{U}_i$ and $\mathcal{U}_j$) by an $d_j \times d_i$-matrix (which can be arbitrary!) Thus, all representations with prescribed dimensions $d_1, \dots, d_n$ of the spaces form themselves a vector space of the total dimension $\sum_{\text{edges } e_{ij}} d_i d_j$ (this is how many entries the matrices A_{ij} have).

Thus, a representation is specified by a collection $\{A_{ij}\}$ of matrices of prescribed sizes $d_j \times d_i$, one for each edge, but some such collections define *equivalent* representations. Which ones? The representation defined by the collection $\{A'_{ij}\}$ is equivalent to the previous one, if there exist invertible transformations $C_i : \mathcal{U}_i \to \mathcal{U}_i$ such that $A'_{ij} = C_j A_{ij} C_i^{-1}$ for each edge e_{ij}. In other words, equivalent representations are obtained from each other by bases changes in the spaces $\mathcal{U}_i = \mathbb{K}^{d_i}$. Such bases changes are determined by n invertible matrices C_i of sizes $d_i \times d_i$, which depend therefore on $\sum_{\text{vertices } v_i} d_i^2$ parameters. In fact one parameter here is "wasted": if all matrices C_i are equal to the same non-zero scalar λ (i.e. $C_i = \lambda I_{d_i}$), then $A'_{ij} = A_{ij}$. We conclude that, ***the number of parameters, needed to parameterize equivalence classes of representations with prescribed dimensions $d_1, \dots, d_n$ of the spaces, is greater than*** $\sum_{\text{edges } e_{ij}} d_i d_j - \sum_{\text{vertices } v_i} d_i^2$.

[2] We refer to [1] for an illuminating proof of Gabriel's theorem is given.

EXERCISES

448. Interpret representations of the two quivers $\tilde{A}_1$ (Figure 47) in matrix terms. ✓

449. On the plane, consider three distinct lines passing through the origin as a representation of quiver D_4 and show that they form a single equivalence class. ϟ

450. Consider four distinct lines on the plane as a representation of quiver $\tilde{D}_4$, and show that the equivalence classes depend on one parameter. ϟ

451.*** Prove that the **cross-ratio** $\lambda := (a-b)(c-d)/(a-c)(b-d)$ of the slopes a, b, c, d of four lines on the plane does not change under linear transformations of the plane. ϟ

452. Find all indecomposable representations of quiver A_1. ✓

453.*** Show directly that equivalence classes of some representations of quiver $\tilde{A}_n$ (with any orientation of the edges) depend on at least one parameter. ϟ

454. Consider a Jordan cell of size 2 as a representation of quiver $\tilde{A}_0$, and find all nontrivial subrepresentations. ✓

455. Show that all complete flags in $\mathbb{K}^n$ considered as representations of quiver A_n are equivalent.

456. Show that pairs of complete flags in $\mathbb{K}^n$ considered as representations of quiver A_{2n-1} form $n!$ equivalence classes. ϟ

Graphs and quadratic forms

To a graph Γ with n vertices $v_1, \ldots, v_n$ and edges $\{e_{ij}\}$, we associate quadratic form Q_Γ on the space $\mathbb{R}^n$ with coordinates $x_1, \ldots x_n$ by the formula

$$Q_\Gamma(\mathbf{x}) := \sum_{\text{vertices } v_i} 2x_i^2 - \sum_{\text{edges } e_{ij}} 2x_i x_j.$$

`Example 13.` When Γ is the graph A_n, the quadratic form is $2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + \cdots + 2x_{n-1}^2 - 2x_{n-1}x_n + 2x_n^2$. The quadratic form corresponding to $\Gamma = \tilde{A}_0$ is $2x_1^2 - 2x_1x_1 = 0$ (identically zero!)

`Proposition.` ***If a quiver Γ is simple, then the quadratic form Q_Γ is positive definite.***

`Proof.` We argue *ad absurdum.* Suppose that $\mathbf{x} \neq 0$, and $Q_\Gamma(\mathbf{x}) \leq 0$. We may assume that all components of $\mathbf{x} = (x_1, \ldots, x_n)$ are rational numbers. Indeed, if Q_Γ is non-negative, but not positive definite, then the coefficient matrix Q_Γ is degenerate. Since it has integer coefficients, the system $Q_\Gamma \mathbf{x} = \mathbf{0}$ has non-trivial *rational* solution. Alternatively, if $Q_\Gamma(\mathbf{x}_0) < 0$ for some $\mathbf{x}_0$, one can approximate $\mathbf{x}_0$ with a rational vector $\mathbf{x}$ and still have $Q_\Gamma(\mathbf{x}) < 0$. Moreover, by clearing denominators, we may assume that all x_i are integers. Note that replacing all x_i with $|x_i|$ can only decrease the value $Q_\Gamma(\mathbf{x})$, because the terms x_i^2 do not change, but the terms $-x_i x_j$, if change at all, then from a positive value to the opposite negative one. Thus, unless Q_Γ is positive definite, there exist a vector $\mathbf{d} = (d_, \dots, d_n)$ of non-negative integers not all equal to 0 and such that $Q_\Gamma(\mathbf{d}) \leq 0$. Therefore equivalence classes of representations of the quiver Γ with the dimensions of the spaces equal to $d_1, \dots, d_n$ (and any orientation of the edges) depend on at least one parameter (on more than $-Q_\Gamma(\mathbf{d})/2$ parameters, to be more precise). Thus the number of such equivalence classes is infinite (since $\mathbb{K}$ is), and the quiver is not simple.

`Theorem.` ***The quadratic form*** Q_Γ ***of a graph*** Γ ***is positive definite if and only if each connected component of the graph is one of:*** $A_n, n \geq 1$***,*** $D_n, n \geq 4$***,*** E_6, E_7, E_8 (Figure 49).

We will prove this theorem in several steps.

Put $\Delta(\Gamma) := \det Q_\Gamma$, the determinant of the coefficient matrix of the quadratic form Q_Γ.

`Proposition.`

$$\Delta(A_n) = n+1,\ \ \Delta(D_n) = 4,\ \ \Delta(E_6) = 3,\ \ \Delta(E_7) = 2,\ \ \Delta(E_8) = 1.$$

`Lemma.` *Let* v_1 *be a vertex of* Γ *connected by one edge with the rest of the graph,* Γ'*, at the vertex* v_2*, and let* Γ'' *be the graph obtained from* Γ' *by removing the vertex* v_2 *together with all the edges attached to it. Then* $\Delta(\Gamma) = 2\Delta(\Gamma') - \Delta(\Gamma'')$.

Indeed, $\det Q_\Gamma$ looks this way, where $\mathbf{0}$ is a row/column of zeroes, and $*$ is a (row/column) of "wild cards":

$$\begin{vmatrix} 2 & -1 & \mathbf{0} \\ -1 & * & * \\ \mathbf{0} & * & Q_{\Gamma''} \end{vmatrix}.$$

Applying the cofactor expansion in the 1st row, and then in the 1st column, we find $\Delta(\Gamma) = 2\Delta(\Gamma') - (-1)^2\Delta(\Gamma'')$, as required. $\square$.

Now, to prove the proposition for $\Gamma = A_n$, we use induction on n. Namely, $\Delta(A_1) = 2$, $\Delta(A_2) = 3$, and from the induction hypothesis $\Delta(A_{n-1}) = n$, $\Delta(A_{n-2}) = n-1$, we conclude using Lemma:

$$\Delta(A_n) = 2\Delta(A_{n-1}) - \Delta(A_{n-2}) = 2n - (n-1) = n+1.$$

For $\Gamma = D_n, E_n$, we apply Lemma to the vertex v_2 with 3 edges, and take v_1 to be the end of the shortest leg. The graph Γ'' falls apart into two components Γ_1, Γ_2. The corresponding matrix is block-diagonal, and the determinant factors: $\Delta(\Gamma'') = \Delta(\Gamma_1)\Delta(\Gamma_2)$. Thus:

$$\Delta(D_n) = 2\Delta(A_{n-1}) - \Delta(A_1)\Delta(A_{n-3}) = 2n - 2(n-2) = 4,$$
$$\Delta(E_6) = 2\Delta(A_5) - \Delta(A_2)\Delta(A_2) = 2\cdot 6 - 3\cdot 3 = 3,$$
$$\Delta(E_7) = 2\Delta(A_6) - \Delta(A_2)\Delta(A_3) = 2\cdot 7 - 3\cdot 4 = 2,$$
$$\Delta(E_8) = 2\Delta(A_7) - \Delta(A_2)\Delta(A_4) = 2\cdot 8 - 3\cdot 5 = 1. \quad \square$$

Corollary 1. ***For*** $\Gamma = A_n, D_n, E_6, E_7, E_8$***, the quadratic form*** Q_Γ ***is positive definite.***

Proof. This follows from Sylvester's rule. Order somehow the vertices of the graph Γ, and tear them off one by one (together with the attached edges). On each step, we obtain a graph Γ' which is one of the graphs A_k, D_k, E_k, or a collection thereof. By Proposition, $\Delta(\Gamma') > 0$. This means that all leading minors of the coefficient matrix Q_Γ are positive, and hence the quadratic form is positive definite. $\square$

Corollary 2. ***For each of the graphs*** $\tilde{\Gamma} := \tilde{A}_n, n \geq 0$***,*** $\tilde{D}_n, n \geq 4$***,*** $\tilde{E}_n$***,*** $n = 6, 7, 8$ ***shown on Figure 50, the corresponding quadratic form*** $Q_{\tilde{\Gamma}}$ ***on*** $\mathbb{R}^{n+1}$ ***is non-negative. More precisely, its positive inertia index equals*** n***, and the 1-dimensional kernel is spanned by the vector whose components are shown on the diagram.***

Proof. By tearing off $\tilde{\Gamma}$ the "white" vertex together with the attached edges, we obtain the corresponding graph Γ, whose quadratic form Q_Γ is positive definite by Corollary 1. Thus, $\mathbb{R}^{n+1}$ contains the subspace $\mathbb{R}^n$ on which the quadratic form $Q_{\tilde{\Gamma}}$ is positive definite, which proves that the positive inertia index is at least n. To prove that the quadratic form $Q_{\tilde{\Gamma}}$ is degenerate, consider the corresponding symmetric bilinear form

$$Q_{\tilde{\Gamma}}(\mathbf{x}, \mathbf{y}) = \sum_{\text{vertices } v_i} (2x_i - \sum_{\text{edges } e_{ij}} x_j) y_i,$$

where the last sum is taken over *all* edges attached to the vertex v_i. To show that a vector $\mathbf{x}$ is Q-orthogonal to every $\mathbf{y}$, it suffices to check that for each vertex v_i, twice the value x_i is equal to the sum of x_j over all vertices v_j connected to v_i: $2x_i = \sum_{\text{edges } e_{ij}} x_j$.

It is straightforward to check that this requirement holds true for the positive integers written next to the vertices on the diagrams of Figure 50. Thus, the vector with these components lies in the quadratic form, which also shows that the negative inertia index of the quadratic form must be 0. □

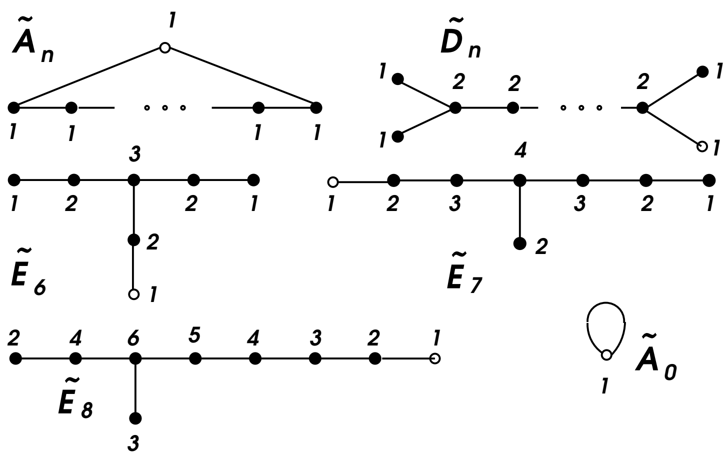

Figure 50

Corollary 2 shows that the quadratic form Q_Γ cannot be positive definite if Γ contains any of the graphs of Figure 50 as a subgraph. (By a *subgraph* of Γ we mean some vertices of Γ connected by *some* of the edges of Γ.) Indeed, the labels on Figure 50 exhibit a vector with non-negative components on which Q_Γ is non-positive. For, adding extra edges can only decrease the value of the form, and the effect of adding extra vertices can be offset by putting $x_i = 0$ at these vertices. Thus, to complete the proof of our Theorem, it remains to show that ***a connected graph*** Γ***, free of subgraphs*** $\tilde{A}_n, n \geq 0$***,*** $\tilde{D}_n, n \geq 4$***, and*** $\tilde{E}_n, n = 6, 7, 8$***, must be one of the graphs*** $A_n, n \geq 1$***,*** $D_n, n \geq 4$***,*** $E_n, n = 6, 7, 8$***.***

To justify the highlighted claim, note that Γ cannot contain a *loop* (i.e. a vertex connected by an edge with itself), nor a *cycle* (for, $\tilde{A}_n$ are exactly these). A connected graph free of loops or cycles is called a *tree*. Thus Γ is a tree. This tree cannot contain a vertex with ≥ 4 edges attached (for, $\tilde{D}_4$ is just that). Nor it can contain two vertices with 3 edges attached to each of them (for, a subgraph $\tilde{D}_n$ with $n > 4$ will be found by connecting these vertices with a chain of edges).

If Γ has no vertices with 3 edges attached, then it is of type A_n. If it has one such a vertex, then $\Gamma = T_{p,q,r}$, i.e. it has the shape of a letter "T" with the legs of type A_p, A_q, and A_r (where p, q, r are integers > 1) connected at a common vertex. If $p, q, r \geq 3$, then Γ contains $\tilde{E}_6$ as a subgraph. If $p = 2$, but $q, r \geq 4$, then Γ contains $\tilde{E}_7$ as a subgraph. If $p = 2, q = 3$, but $r \geq 6$, then Γ contains $\tilde{E}_8$ as a subgraph. Thus, assuming $p \leq q \leq r$, we have only the following options left: $(p, q, r) = (2, 3, 5)$, $(2, 3, 4)$, $(2, 3, 3)$, which yield the graphs E_8, E_7, E_6, or $p = q = 2$ (while $r \geq 2$ can be arbitrary), which yields the graphs D_{r+2}. Theorem is proved.

EXERCISES

457.* Prove that $\Delta(T_{p,q,r}) = pq - qr + rp - pqr$. ↯

458. Show that $Q_{T_{p,q,r}}$ is positive definite (respectively, non-negative) if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (respectively $= 1$). ↯

459.* Find all integer triples (p, q, r), $2 \leq p \leq q \leq r$, satisfying
(a) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$; (b) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. ✓

460. Tile the Euclidean plane by congruent triangles with the angles:
(a) $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, (b) $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, (c) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$.

461.* Tile the sphere by spherical triangles with the angles:
(a) $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{r})$; (b) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$, (c) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})$, (d) $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5})$. ↯

Root Systems

A remarkable feature of the classification by the graphs A_n, D_n, E_n is that they arise not only in connection with quivers, or graphs Γ with positive definite quadratic forms Q_Γ, but in a myriad of other classification problems. Namely, they occur in the theories of: regular polyhedra in the 3-space, reflection groups in Euclidean spaces, compact Lie groups, degenerations of functions near critical points, singularities of wave fronts and caustics of geometrical optics, and perhaps in many others. There exist many direct connections between different manifestations of the ADE-classification, but the general cause of the phenomenon remains a mystery. We round up these notes with one illustration to the mystery, which comes from the theory of reflection groups.

For an introduction to group theory we refer the reader to any of the items [5, 9, 11] in our short bibliography, but in what follows we try to describe the groups we need directly with no reference to the general theory.

Given a graph Γ, one can consider the symmetric bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle = (Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}))/2$ associated with the quadratic form $Q = Q_\Gamma$ as an inner product in $\mathbb{R}^n$ in a generalized sense (for it is not guaranteed to be positive definite). Let $\mathbf{v}_i$ be the vectors of the standard basis in $\mathbb{R}^n$. They correspond to the vertices of the graph and have length $\sqrt{2}$, i.e. $Q(\mathbf{v}_i) = 2$ (assuming that Γ has no loops attached to the ith vertex). One can associate to Γ the *group* G_Γ generated by n reflections in the hyperplanes Q-orthogonal to $\mathbf{v}_i$.

In more detail, by a **group** of Q-orthogonal transformations one means a collection $G = \{U_\alpha\}$ of transformations $U_\alpha : \mathbb{R}^n \to \mathbb{R}^n$, preserving the inner product, i.e. satisfying $\langle U_\alpha \mathbf{x}, U_\alpha \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and such that the compositions $U_\alpha U_\beta$ and inverses U_α^{-1} of the transformations from the collection G are also in G.

For instance, to each vector $\mathbf{v} \in \mathbb{R}^n$ with $Q(\mathbf{v}) = 2$ one can associate the **reflection** $R_\mathbf{v}$ in the hyperplane Q-orthogonal to $\mathbf{v}$:

$$R_\mathbf{v}\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{v} \rangle \mathbf{v}.$$

If $\langle \mathbf{x}, \mathbf{v} \rangle = 0$, then $R_\mathbf{v}\mathbf{x} = \mathbf{x}$, while $R_\mathbf{v}\mathbf{v} = \mathbf{v} - 2\mathbf{v} = -\mathbf{v}$. Thus $\mathbb{R}^n$ is the direct Q-orthogonal sum of two eigenspaces of $R_\mathbf{v}$ corresponding to the eigenvalues 1 and -1, and so $R_\mathbf{v}$ preserves Q.

Given several vectors $\{\mathbf{v}_i\}$ with $Q(\mathbf{v}_i) = 2$, one can generate a group of Q-orthogonal transformations by considering all the reflections $R_{\mathbf{v}_i}$ along with their compositions, inverses, compositions of the compositions, etc.

It turns out that ***the reflection group*** G_Γ ***associated with a graph*** Γ ***without loops is finite if and only if each connected component of*** Γ ***is one of the graphs*** A_n, D_n, E_6, E_7, E_8***.*** We are not going to prove this theorem here (although this would not be too hard to do by checking that if Γ contains $\tilde{A}_n, \tilde{D}_n$, or $\tilde{E}_n$, then the reflection group must be infinite), but merely illustrate it with a useful example.

`Example 14:` *Reflection group* A_n. In the standard Euclidean space $\mathbb{R}^{n+1}$ with coordinates $x_0, \dots, x_n$, consider the hyperplane given by the linear equation $x_0 + \cdots + x_n = 0$. Permutations of the coordinates form a group of $(n+1)!$ orthogonal transformations in $\mathbb{R}^{n+1}$, preserving the hyperplane. The whole group of permutations is generated by n transpositions $\tau_{01}, \tau_{12}, \dots, \tau_{n-1,n}$, where τ_{ij} stands for the swapping of the coordinates x_i and x_j, and thus acts as an orthogonal reflection in the hyperplane $x_i = x_j$.

The hyperplane $x_0 + \cdots + x_n = 0$ can be identified with $\mathbb{R}^n$ by the choice of a basis:

$$\mathbf{v}_1 = \mathbf{e}_0 - \mathbf{e}_1,\ \mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2,\ \ldots,\ \mathbf{v}_n = \mathbf{e}_{n-1} - \mathbf{e}_n,$$

where $\mathbf{e}_i = (\ldots, 0, 1, 0, \ldots)^t$ denote the unit coordinate vectors in $\mathbb{R}^{n+1}$. Computing pairwise dot-products, we find:

$$\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 2,\ \langle \mathbf{v}_i, \mathbf{v}_{i+1} \rangle = -1,\ \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0 \text{ for } |i - j| > 1.$$

We see that the Euclidean structure on the hyperplane $\mathbb{R}^n$ in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ coincides with the one defined by the graph A_n. Note that the reflections in the hyperplanes perpendicular to $\mathbf{v}_i$ are exactly the transpositions $\tau_{i-1,i}$. Thus, the reflection group G_{A_n} is identified with the group of permutations of $n+1$ objects.

There is more here than meets the eye. While the group is *generated* by n reflections, the total number of hyperplane reflections in it is $\binom{n+1}{2} = n(n+1)/2$: one for each transposition τ_{ij}. Respectively there are $n(n+1)$ vectors $\mathbf{v}$ of length $\sqrt{2}$ perpendicular to the hyperplanes: $\pm(\mathbf{e}_i - \mathbf{e}_j)$. The configuration of these vectors (called "roots") is called the **root system** associated to the graph A_n. The same can be done with each of the A, D, E-graphs: the root system is a finite symmetric configuration of vectors of length $\sqrt{2}$ obtained from any of the vectors $\mathbf{v}_i$ by applying all transformations from the reflection group. One of many peculiar properties of root systems is that *each root* $\mathbf{v}$ *is a linear combination of the basis* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *(corresponding to the vertices of the graph) with coefficients which have the same sign: either all non-negative, or all non-positive.* E.g. in the case A_n, if $i < j$, then $\pm(\mathbf{e}_i - \mathbf{e}_j) = \pm(\mathbf{v}_{i+1} + \cdots + \mathbf{v}_j)$. Thus, all the roots are divided into positive (including $\mathbf{v}_1, \ldots, \mathbf{v}_n$) and negative.

We can now state the following addition to Gabriel's theorem:

Indecomposable representations of a simple quiver are in one-to-one correspondence with positive roots of the corresponding root system, and the dimension of the vector space associated with a vertex of the quiver in a given indecomposable representation coincides with the coefficient $d_i \geq 0$ ***of the corresponding positive root*** $\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n$ ***at the simple root*** $\mathbf{v}_i$ ***associated with this vertex.***

`Example 15:` A_2. The reflection group G_{A_2} acts on the plane $x_0 + x_1 + x_3 = 0$ by symmetries of a regular triangle. There are three symmetry axes, and respectively 6 roots perpendicular to them, three of which are positive: $\mathbf{v}_1 = \mathbf{e}_0 - \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_1 - \mathbf{e}_2$, and $\mathbf{v} = \mathbf{e}_0 - \mathbf{e}_2 = \mathbf{v}_1 + \mathbf{v}_2$. Their coefficients $(1,0)$ $(0,1)$, and $(1,1)$ correspond to the three indecomposable representations of the quiver $\bullet \to \bullet$ as described by the Rank Theorem: $\mathbb{K}^1 \to \mathbb{K}^0$, $\mathbb{K}^0 \to \mathbb{K}^1$, and $\mathbb{K}^1 \stackrel{\cong}{\to} \mathbb{K}^1$.

EXERCISES

462. Verify that reflections $R_{\mathbf{v}}$ are Q-orthogonal. ✓

463.* Prove that the reflection group G_Γ corresponding to the graph $\Gamma = \tilde{A}_1$ is infinite. ϟ

464. Describe all indecomposable representations of quiver A_n: $\bullet \to \bullet \to \cdots \to \bullet \to \bullet$. ϟ

465. Represent a complete flag in $\mathbb{K}^n$ as the direct sum of indecomposable representations of quiver A_n. ✓

466. Let X denote an $n \times n$-matrix. Find the solution to the ODE system $dX/dt = AX - XA$ (where X stands for the unknown $n \times n$-matrix), given the initial value $X(0)$. ✓

467. In the previous exercise, let A be diagonal, with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvectors and eigenvalues of the operator $X \mapsto AX - XA$ on the space of $n \times n$-matrices, and compare the answer with the root system of type A. ✓

Index

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