397. Construct a triangle, given the ratio of its altitude to the base, the angle at the vertex, and the median drawn to one of its lateral sides

398. Into a given disk segment, inscribe a square such that one of its sides lies on the chord, and the opposite vertices on the arc.

399. Into a given triangle, inscribe a rectangle with the given ratio of the sides m : n, so that one of its sides lies on the base of the triangle, and the opposite vertices on the lateral sides.

6 Geometric mean

184. Definition. The geometric mean between two segments a and c is defined to be a third segment b such that a : b = b : c. More generally, the same definition applies to any quantities of the same denomination. When a, b, and c are positive numbers, the relationship a : b = b : c can be rewritten as

$$b^2 = ac$$
, or $b = \sqrt{ac}$.

185. Theorem. In a right triangle:

(1) the altitude dropped from the vertex of the right angle is the geometric mean between two segments into which the foot of the altitude divides the hypotenuse, and

(2) each leg is the geometric mean between the hypotenuse and the segment of it which is adjacent to the leg.

Let AD (Figure 188) be the altitude dropped from the vertex of the right angle A to the hypotenuse BC. It is required to prove the following proportions:

(1)
$$\frac{BD}{AD} = \frac{AD}{DC}$$
, (2) $\frac{BC}{AB} = \frac{AB}{BD}$ and $\frac{BC}{AC} = \frac{AC}{DC}$.

The first proportion is derived from similarity of the triangles BDA and ADC. These triangles are similar because

$$\angle 1 = \angle 4$$
 and $\angle 2 = \angle 3$

as angles with perpendicular respective sides (§80). The sides BD and AD of $\triangle BDA$ form the first ratio of the required proportion.

The homologous sides of $\triangle ADC$ are AD and DC, ³ and therefore BD: AD = AD: DC.

The second proportion is derived from similarity of the triangles ABC and BDA. These triangles are similar because both are right, and $\angle B$ is their common acute angle. The sides BC and AB of $\triangle ABC$ form the first ratio of the required proportion. The homologous sides of $\triangle BDA$ are AB and BD, and therefore BC : AB = AB : BD.

The last proportion is derived in the same manner from the similarity of the triangles ABC and ADC.



186. Corollary. Let A (Figure 189) be any point on a circle, described about a diameter BC. Connecting this point by chords with the endpoints of the diameter we obtain a right triangle such that its hypotenuse is the diameter, and its legs are the chords. Applying the theorem to this triangle we arrive at the following conclusion:

The perpendicular dropped from any point of a circle to its diameter is the geometric mean between the segments into which the foot of the perpendicular divides the diameter, and the chord connecting this point with an endpoint of the diameter is the geometric mean between the diameter and the segment of it adjacent to the chord.

187. Problem. To construct the geometric mean between two segments a and c.

We give two solutions.

(1) On a line (Figure 190), mark segments AB = a and BC = c next to each other, and describe a semicircle on AC as the diameter.

³In order to avoid mistakes in determining which sides of similar triangles are homologous to each other, it is convenient to mark angles opposite to the sides in question of one triangle, then find the angles congruent to them in the other triangle, and then take the sides opposite to these angles. For instance, the sides *BD* and *AD* of $\triangle BDA$ are opposite to the angles 1 and 3; these angles are congruent to the angles 4 and 2 of $\triangle ADC$, which are opposite to the sides *AD* and *DC*. Thus the sides *AD* and *DC* correspond to *BD* and *AD* respectively.

From the point B, erect the perpendicular to AC up to the intersection point D with the semicircle. The perpendicular BD is the required geometric mean between AB and BC.



(2) From the endpoint A of a ray (Figure 191), mark the given segments a and b. On the greater of them, describe a semicircle. From the endpoint of the smaller one, erect the perpendicular up to the intersection point D with the semicircle, and connect D with A. The chord AD is the required geometric mean between a and b.

188. The Pythagorean Theorem. The previous theorems allow one to obtain a remarkable relationship between the sides of any right triangle. This relationship was proved by the Greek geometer *Pythagoras of Samos* (who lived from about 570 B.C.to about 475 B.C.) and is named after him.

Theorem. If the sides of a right triangle are measured with the same unit, then the square of the length of its hypotenuse is equal to the sum of the squares of the lengths of its legs.



Figure 192

Let ABC (Figure 192) be a right triangle, and AD the altitude dropped to the hypotenuse from the vertex of the right angle. Suppose that the sides and the segments of the hypotenuse are measured by the same unit, and their lengths are expressed by the numbers a, b, c, c' and b'.⁴ Applying the theorem of §185, we obtain the proportions:

$$a: c = c: c'$$
 and $a: b = b: b'$,

or equivalently:

$$ac' = c^2$$
 and $ab' = b^2$.

Adding these equalities, we find:

$$ac' + ab' = c^2 + b^2$$
, or $a(c' + b') = c^2 + b^2$.

But c' + b' = a, and therefore $a^2 = b^2 + c^2$.

This theorem is often stated in short: the square of the hypotenuse equals the sum of the squares of the legs.

Example. Suppose that the legs measured with some linear unit are expressed by the numbers 3 and 4. Then the hypotenuse is expressed in the same units by a number x such that

 $x^{2} = 3^{2} + 4^{2} = 9 + 16 = 25$, and hence $x = \sqrt{25} = 5$.

Remark. The right triangle with the sides 3, 4, and 5 is sometimes called *Egyptian* because it was known to ancient Egyptians. It is believed they were using this triangle to construct right angles on the land surface in the following way. A circular rope marked by 12 knots spaced equally would be stretched around three poles to form a triangle with the sides of 3, 4, and 5 spacings. Then the angle between the sides equal to 3 and 4 would turn out to be right. ⁵

Yet another formulation of the Pythagorean theorem, namely the one known to Pythagoras himself, will be given in §259.

189. Corollary. The squares of the legs have the same ratio as the segments of the hypotenuse adjacent to them.

Indeed, from formulas in §188 we find $c^2: b^2 = ac': ab' = c': b'$. Remarks. (1) The three equalities

$$ac'=c^2, \ ab'=b^2, \ a^2=b^2+c^2,$$

⁴It is customary to denote sides of triangles by the lowercase letters corresponding to the uppercase letters which label the opposite vertices.

⁵Right triangles whose sides are measured by *whole* numbers are called **Pythagorean**. One can prove that the legs x and y, and the hypotenuse z of such triangles are expressed by the formulas: $x = 2ab, y = a^2 - b^2, z = a^2 + b^2$, where a and b are arbitrary whole numbers such that a > b.

can be supplemented by two more:

$$b' + c' = a$$
, and $h^2 = b'c'$,

where h denotes the length of the altitude AD (Figure 192). The third of the equalities, as we have seen, is a consequence of the first two and of the fourth, so that only four of the five equalities are independent. As a result, given two of the six numbers a, b, c, b', c' and h, we can compute the remaining four. For example, suppose we are given the segments of the hypotenuse b' = 5 and c' = 7. Then

$$a = b' + c' = 12, \ c = \sqrt{ac'} = \sqrt{12 \cdot 7} = \sqrt{84} = 2\sqrt{21},$$

 $b = \sqrt{ab'} = \sqrt{12 \cdot 5} = \sqrt{60}, \ h = \sqrt{b'c'} = \sqrt{5 \cdot 7} = \sqrt{35}.$

(2) Later on we will often say: "the square of a segment" instead of "the square of the number expressing the length of the segment," or "the product of segments" instead of "the product of numbers expressing the lengths of the segments." We will assume therefore that all segments have been measured using the same unit of length.

190. Theorem. In every triangle, the square of a side opposite to an acute angle is equal to the sum of the squares of the two other sides minus twice the product of (any) one of these two sides and the segment of this side between the vertex of the acute angle and the foot of the altitude drawn to this side.

Let BC be the side of $\triangle ABC$ (Figures 193 and 194), opposite to the acute angle A, and BD the altitude dropped to another side, e.g. AC, (or to its extension). It is required to prove that

$$BC^2 = AB^2 + AC^2 - 2AC \cdot AD,$$

or, using the notation of the segments by single lowercase letters as shown on Figures 193 or 194, that

$$a^2 = b^2 + c^2 - 2bc'.$$

From the right triangle BDC, we have:

$$a^2 = h^2 + (a')^2. \tag{(*)}$$

Let us compute each of the squares h^2 and $(a')^2$. From the right triangle BAD, we find: $h^2 = c^2 - (c')^2$. On the other hand, a' = b - c'

(Figure 193) or a' = c' - b (Figure 194). In both cases we obtain the same expression for $(a')^2$:

$$(a')^2 = (b - c')^2 = (c' - b)^2 = b^2 - 2bc' + (c')^2.$$

Now the equality (*) can be rewritten as

$$a^{2} = c^{2} - (c')^{2} + b^{2} - 2bc' + (c')^{2} = c^{2} + b^{2} - 2bc'.$$



191. Theorem. In an obtuse triangle, the square of the side opposite to the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of (any) one of these two sides and the segment on the extension of this side between the vertex of the obtuse angle and the foot of the altitude drawn to this side.

Let AB be the side of $\triangle ABC$ (Figure 194), opposite to the obtuse angle C, and BD the altitude dropped to the extension of another side, e.g. AC. It is required to prove that

$$AB^2 = AC^2 + BC^2 + 2AC \cdot CD,$$

or, using the abbreviated notation shown in Figure 194, that

$$c^2 = a^2 + b^2 + 2ba'.$$

From the right triangles ABD and CBD, we find:

$$c^{2} = h^{2} + (c')^{2} = a^{2} - (a')^{2} + (a'+b)^{2} = a^{2} - (a')^{2} + (a')^{2} + 2ba' + b^{2} = a^{2} + b^{2} + 2ba'.$$

192. Corollary. From the last three theorems, we conclude, that the square of a side of a triangle is equal to, greater than, or smaller than the sum of the squares of the other two sides, depending on whether the angle opposite to this side is right, acute, or obtuse.

Furthermore, this implies the converse statement: an angle of a triangle turns out to be right, acute or obtuse, depending on whether the square of the opposite side is equal to, greater than, or smaller than the sum of the squares of the other two sides.

193. Theorem. The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides (Figure 195).



Figure 195

From the vertices B and C of a parallelogram ABCD, drop the perpendiculars BE and CF to the base AD. Then from the triangles ABD and ACD, we find:

 $BD^2 = AB^2 + AD^2 - 2AD \cdot AE, \quad AC^2 = AD^2 + CD^2 + 2AD \cdot DF.$

The right triangles ABE and DCF are congruent, since they have congruent hypotenuses and congruent acute angles, and hence AE = DF. Having noticed this, add the two equalities found earlier. The summands $-2AD \cdot AE$ and $+2AD \cdot DF$ cancel out, and we get:

 $BD^2 + AC^2 = AB^2 + AD^2 + AD^2 + CD^2 = AB^2 + BC^2 + CD^2 + AD^2.$

194. We return to studying geometric means in a disk.

Theorem. If through a point (M, Figure 196), taken inside a disk, a chord (AB) and a diameter (CD) are drawn, then the product of the segments of the chord $(AM \cdot MB)$ is equal to the product of the segments of the diameter $(CM \cdot MD)$.

Drawing two auxiliary chords AC and BD, we obtain two triangles AMC and DMB (shaded in Figure 196) which are similar,

since their angles A and D are congruent as inscribed intercepting the same arc BC, and the angles B and D are congruent as inscribed intercepting the same arc AD. From similarity of the triangles we derive: AM : MD = CM : MB, or equivalently



$$AM \cdot MB = CM \cdot MD.$$

195. Corollaries. (1) For all chords (AB, EF, KL, Figure 196) passing through the same point (M) inside a disk, the product of the segments of each chord is constant, i.e. it is the same for all such chords, since for each chord it is equal to the product of the segments of the diameter.

(2) The geometric mean between the segments (AM and MB) of a chord (AB), passing through a point (M) given inside a disk, is the segment (EM or MF) of the chord (EF) perpendicular to the diameter (CD), at the given point, because the chord perpendicular to the diameter is bisected by it, and hence

$$EM = MF = \sqrt{AM \cdot MB}.$$

196. Theorem. The tangent (MC, Figure 197) from a point (M) taken outside a disk is the geometric mean between a secant (MA), drawn through the same point, and the exterior segment of the secant (MB).

Draw the auxiliary chords AC and BC, and consider two triangles MCA and MCB (shaded in Figure 197). They are similar because $\angle M$ is their common angle, and $\angle MCB = \angle BAC$ since each of them

is measured by a half of the arc BC. Taking the sides MA and MC in $\triangle MCA$, and the homologous sides MC and MB in $\triangle MCB$, we obtain the proportion: MA : MC = MC : MB and conclude, that the tangent MC is the geometric mean between the segments MA and MB of the secant.

197. Corollaries. (1) The product of a secant (MA, Figure 197), passing through a point (M) outside a disk, and the exterior part of the secant (MB) is equal to the square of the tangent (MC) drawn from the same point, i.e.:

$$MA \cdot MB = MC^2.$$

(2) For all secants (MA, MD, ME, Figure 197), drawn from a point (M) given outside a disk, the product of each secant and the exterior segment of it, is constant, i.e. the product is the same for all such secants, because for each secant this product is equal to the square MC^2 of the tangent drawn from the point M.

198. Theorem. The product of the diagonals of an inscribed quadrilateral is equal to the sum of the products of its opposite sides.

This proposition is called **Ptolemy's theorem** after a Greek astronomer *Claudius Ptolemy* (85 – 165 A.D.) who discovered it.



Let AC and BD be the diagonals of an inscribed quadrilateral ABCD (Figure 198). It is required to prove that

$$AC \cdot BD = AB \cdot CD + BC \cdot AD.$$

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Construct the angle BAE congruent to $\angle DAC$, and let E be the intersection point of the side AE of this angle with the diagonal BD. The triangles ABE and ADC (shaded in Figure 198) are similar, since their angles B and C are congruent (as inscribed intercepting the same arc AD), and the angles at the common vertex A are congruent by construction. From the similarity, we find:

$$AB: AC = BE: CD$$
, i.e. $AC \cdot BE = AB \cdot CD$.

Consider now another pair of triangles, namely $\triangle ABC$ and $\triangle AED$ (shaded in Figure 199). They are similar, since their angles BAC and DAE are congruent (as supplementing to $\angle BAD$ the angles congruent by construction), and the angles ACB and ADB are congruent as inscribed intercepting the same angle AB. We obtain:

$$BC: ED = AC: AD$$
, i.e. $AC \cdot ED = BC \cdot AD$.

Summing the two equality, we find:

 $AC(BE + ED) = AB \cdot CD + BC \cdot AD$, where BE + ED = BD.

EXERCISES

Prove theorems:

400. If a diagonal divides a trapezoid into two similar triangles, then this diagonal is the geometric mean between the bases.

 $401.^{*}$ If two disks are tangent externally, then the segment of an external common tangent between the tangency points is the geometric mean between the diameters of the disks.

402. If a square is inscribed into a right triangle in such a way that one side of the square lies on the hypotenuse, then this side is the geometric mean between the two remaining segments of the hypotenuse.

403.[★] If AB and CD are perpendicular chords in a circle of radius R, then $AC^2 + BD^2 = 4R^2$.

404. If two circles are concentric, then the sum of the squares of the distances from any point of one of them to the endpoints of any diameter of the other, is a fixed quantity.

Hint: See §193.

405. If two segments AB and CD (or the extensions of both segments) intersect at a point E, such that $AE \cdot EB = CE \cdot ED$, then the points A, B, C, D lie on the same circle.

Hint: This is the theorem converse to that of §195 (or §197).

406.* In every $\triangle ABC$, the bisector AD satisfies $AD^2 = AB \cdot AC - DB \cdot DC$.

Hint: Extend the bisector to its intersection E with the circumscribed circle, and prove that $\triangle ABD$ is similar to $\triangle AEC$.

407.* In every triangle, the ratio of the sum of the squares of all medians to the sum of the squares of all sides is equal to 5/4.

408. If an isosceles trapezoid has bases a and b, lateral sides c, and diagonals d, then $ab + c^2 = d^2$.

409. The diameter AB of a circle is extended past B, and at a point C on this extension $CD \perp AB$ is erected. If an arbitrary point M of this perpendicular is connected with A, and the other intersection point of AM with the circle is denoted A', then $AM \cdot AA'$ is a fixed quantity, i.e. it does not depend on the choice of M.

410.^{*} Given a circle \mathcal{O} and two points A and B. Through these points, several circles are drawn such that each of them intersects with or is tangent to the circle \mathcal{O} . Prove that the chords connecting the intersection points of each of these circles, as well as the tangents at the points of tangency with the circle \mathcal{O} , intersect (when extended) at one point lying on the extension of AB.

411. Using the result of the previous problem, find a construction of the circle passing through two given points and tangent to a given circle.

Find the geometric locus of:

412. Points for which the sum of the squares of the distances to two given points is a fixed quantity.

Hint: See §193.

413. Points for which the difference of the squares of the distances from two given points is a fixed quantity.

Computation problems

414. Compute the legs of a right triangle if the altitude dropped from the vertex of the right angle divides the hypotenuse into two segments m and n.

415. Compute the legs of a right triangle if a point on the hypotenuse equidistant from the legs divides the hypotenuse into segments 15 and 20 cm long.

416. The centers of three pairwise tangent circles are vertices of a right triangle. Compute the smallest of the three radii if the other two are 6 and 4 cm.