## 5 The Pythagorean theorem revisited

259. Theorem. The areas of squares constructed on the legs of a right triangle add up to the area of the square constructed on its hypotenuse.

This proposition is yet another form of the Pythagorean theorem, which we proved in §188: the square of the number measuring the length of the hypotenuse is equal to the sum of the squares of the numbers measuring the legs. Indeed, the square of the number measuring the length of a segment is the number measuring the area of the square constructed on this segment.

There are many other ways to prove the Pythagorean theorem.

Euclid's proof. Let ABC (Figure 264) be a right triangle, and BDEA, AFGC, and BCKH squares constructed on its legs and the hypotenuse. It is required to prove that the areas of the first two squares add up to the area of the third one.



Draw  $AM \perp BC$ . Then the square BCKH is divided into two rectangles. Let us prove that the rectangle BLMH is equivalent to the square BDEA, and the rectangle LCKM is equivalent to the square AFGC. For this, consider two triangles shaded in Figure 264. These triangles are congruent, since  $\triangle ABH$  is obtained from  $\triangle DBC$  by clockwise rotation about the point B through the angle

of 90°. Indeed, rotating this way the segment BD, which is a side of the square BDEA, we obtain another side BA of this square, and rotating the segment BC, which is a side of the square BCKH, we obtain BH. Thus  $\triangle ABH$  and  $\triangle DBC$  are equivalent. On the other hand,  $\triangle DBC$  has the base DB, and the altitude congruent to BA(since AC||DB). Therefore  $\triangle DBC$  is equivalent to a half of the square BDEA. Likewise,  $\triangle ABH$  has the base BH, and the altitude congruent to BL (since AL||BH). Therefore  $\triangle ABH$  is equivalent to a half of the rectangle BLMH. Thus the rectangle BLMH is equivalent to the square BDEA. Similarly, connecting G with B, and A with K, and considering  $\triangle GCB$  and  $\triangle ACK$ , we prove that the rectangle LCKM is equivalent to the square AFGC. This implies that the square BCKH is equivalent to the sum of the squares BDEA and AFGC.

A tiling proof, shown in Figure 265, is based on tiling the square, whose side is congruent to the sum of the legs of a given right triangle, by the square constructed on the hypotenuse and by four copies of the given triangle, and then re-tiling it by the squares constructed on the legs and by the same four triangles.

One more proof, based on similarity, will be explained shortly.

**260.** Generalized Pythagorean theorem. The following generalization of the Pythagorean theorem is found in the 6th book of Euclid's "Elements."

Theorem. If three similar polygons  $(\mathbf{P}, \mathbf{Q}, \text{ and } \mathbf{R}, \text{ Figure} 266)$  are constructed on the sides of a right triangle, then the polygon constructed on the hypotenuse is equivalent to the sum of the polygons constructed on the legs.

In the special case when the polygons are squares, this proposition turns into the Pythagorean theorem as stated in §259. Due to the theorem of §251, the generalization follows from this special case. Indeed, the areas of similar polygons are proportional to the squares of homologous sides, and therefore

$$\frac{\text{area of } \mathbf{P}}{a^2} = \frac{\text{area of } \mathbf{Q}}{b^2} = \frac{\text{area of } \mathbf{R}}{c^2}.$$

Then, by properties of proportions,

$$\frac{\text{area of } \mathbf{P} + \text{area of } \mathbf{Q}}{a^2 + b^2} = \frac{\text{area of } \mathbf{R}}{c^2}.$$

Since  $a^2 + b^2 = c^2$ , it follows that

area of 
$$\mathbf{P}$$
 + area of  $\mathbf{Q}$  = area of  $\mathbf{R}$ .

Moreover, the same reasoning applies to similar figures more general than polygons. However, Euclid gives another proof of the generalized Pythagorean theorem, which does not rely on this special case. Let us explain such a proof here. In particular, we will obtain one more proof of the Pythagorean theorem itself.



First, let us notice that to prove the generalized Pythagorean theorem, it suffices to prove it for polygons of one shape only. Indeed, suppose that the areas of two polygons  $\mathbf{R}$  and  $\mathbf{R}'$  of different shapes constructed on some segment (e.g. the hypotenuse) have a certain ratio k. Then the areas of polygons similar to them (e.g.  $\mathbf{P}$  and  $\mathbf{P}'$ , or  $\mathbf{Q}$  and  $\mathbf{Q}'$ ) and constructed on another segment which is, say, m times shorter, will be  $m^2$  times smaller for both shapes. Therefore they will have the same ratio k. Thus, if the areas of  $\mathbf{P}'$ ,  $\mathbf{Q}'$  and  $\mathbf{R}'$  satisfy the property that the first two add up to the third one, then the same holds true for the areas of  $\mathbf{P}$ ,  $\mathbf{Q}$  and  $\mathbf{R}$  which are k times greater.

Now the idea is to take polygons similar not to a square, but to the right triangle itself, and to construct them not outside the triangle but inside it.<sup>4</sup>

Namely, drop the altitude of the right triangle to its hypotenuse. The altitude *divides* the triangle into two triangles similar to it. Together with the original triangle, we thus have three similar right triangles constructed on the sides of it, and such that two of the areas add up to the third one.

Corollary. If outside of a right triangle (Figure 267) two semicircles are described on its legs, and another semicircle is described

<sup>&</sup>lt;sup>4</sup>The collage on the cover of this book illustrates this idea.

on the hypotenuse so that it contains the triangle, then the geometric figure bounded by the semicircles is equivalent to the triangle:

area of  $\mathbf{A}$  + area of  $\mathbf{B}$  = area of  $\mathbf{C}$ .

Indeed, after adding to both sides of this equality the areas (unshaded in Figure 267) of the disk segments bounded by the greatest of the semicircles and by the legs of the triangle, it is required to prove that the areas of the half-disks constructed on the legs add up to the area of the half-disk constructed on the hypotenuse. This equality follows from the generalized Pythagorean theorem.

**Remark.** The figures **A** and **B** are known as **Hippocrates' lunes** after a Greek mathematician *Hippocrates of Chios* who studied them in the 5th century B.C. in connection with the problem of squaring the circle. When the triangle is isosceles, then the lunes are congruent and each is equivalent to a half of the triangle.

## EXERCISES

## Miscellaneous problems

**576.** The altitude dropped to the hypotenuse divides a given right triangle into smaller triangles whose radii of the inscribed circles are 6 and 8 cm. Compute the radius of the inscribed circle of the given triangle.

**577.** Compute the sides of a right triangle given the radii of its circumscribed and inscribed circle.

**578.** Compute the area of a right triangle if the foot of the altitude dropped to the hypotenuse of length c divides it in the extreme and mean ratio.

**579.** Compute the area of the quadrilateral bounded by the four bisectors of the angles of a rectangle with the sides a and b cm.

580.<sup>\*</sup> Cut a given rectangle into four right triangles so that they can be reassembled into two smaller rectangles similar to the given one.

**581.** The diagonals divide a quadrilateral into four triangles of which three have the areas 10, 20, and 30  $cm^2$ , and the area of the fourth one is greater. Compute the area of the quadrilateral.

**582.** A circle of the radius congruent to the altitude of a given isosceles triangle is rolling along the base. Show that the arc length cut out on the circle by the lateral sides of the triangle remains constant. **583.** A circle is divided into four arbitrary arcs, and the midpoints of the arcs are connected pairwise by straight segments. Prove that two of the segments are perpendicular.

**584.** Compute the length of a common tangent of two circles of radii r and 2r which intersect at the right angle.

**585.** Prove that in a triangle, the altitudes  $h_a$ ,  $h_b$ ,  $h_c$ , and the radius of the inscribed circle satisfy the relation:  $1/h_a + 1/h_b + 1/h_c = 1/r$ . **586.** Prove that in a right triangle, the sum of the diameters of the inscribed and circumscribed circles is congruent to the sum of the legs.

587.\* Prove that in a scalene triangle, the sum of the diameters of the inscribed and circumscribed circle is congruent to the sum of the segments of the altitudes from the orthocenter to the vertices.

 $588.^{\star}$  Find the geometric locus of all points with a fixed difference of the distances from the sides of a given angle.

589.\* A side of a square is the hypotenuse of a right triangle situated in the exterior of the square. Prove that the bisector of the right angle of the triangle passes through the center of the square, and compute the distance between the center and the vertex of the right angle of the triangle, given the sum of its legs.

 $590.^{\star}$  From each of the two given points of a given line, both tangents to a given circle are drawn, and in the two angles thus formed, congruent circles are inscribed. Prove that their line of centers is parallel to the given line.

**591.**\* Three congruent circles intersect at one point. Prove that the three lines, each passing through the center of one of the circles and the second intersection point of the other two circles, are concurrent.

**592.**<sup>\*</sup> Given a triangle ABC, find the geometric locus of points M such that the triangles ABM and ACM are equivalent.

**593.**<sup>\*</sup> On a given circle, find two points, A and B, symmetric about a given diameter CD and such that a given point E on the diameter is the orthocenter of the triangle ABC.

**594.**<sup>\*</sup> Find the geometric locus of the points of intersection of two chords AC and BD of a given circle, where AB is a fixed chord of this circle, and CD is any chord of a fixed length.

 $595.^{\star}$  Construct a triangle, given its altitude, bisector and median drawn from the same vertex.

596.<sup>\*</sup> Construct a triangle, given its circumcenter, incenter, and the intersection point of the extension of one of the bisectors with the circumscribed circle.