133.[*] Prove that if a figure has two symmetry planes making an angle $180°/n$, then their intersection line is an axis of symmetry of the nth order.

134. Describe the cross section of a cube by the plane perpendicular to one of the diagonals at its midpoint.

135.[*] Show that the 24 ways of superimposing the cube onto itself correspond to 24 different ways (including the trivial one) of permuting its four diagonals.

5   Regular polyhedra

84. Definition. Let us call a polyhedral angle **regular** if all of its plane angles are congruent and all of its dihedral angles are congruent. A polyhedron is called **regular** if all of its faces are congruent regular polygons, and all of its polyhedral angles are regular and congruent. Thus a cube is a regular polyhedron. It follows from the definition, that in a regular polyhedron: all plane angles are congruent, all dihedral angles are congruent, and all edges are congruent.

85. Classification of regular polyhedra. Let us take into account that a convex polyhedral angle has at least three plane angles, and that their sum has to be smaller than $4d$ (§48).

Since in a regular triangle, every angle is $\frac{2}{3}d$, repeating it 3, 4, or 5 times, we obtain the angle sum smaller than $4d$, but repeating it 6 or more times, we get the angle sum equal to or greater than $4d$. Therefore convex polyhedral angles whose faces are angles of regular triangles can be of only three types: trihedral, tetrahedral, or pentahedral. Angles of squares and regular pentagons are respectively d and $\frac{6}{5}d$. Repeating these angles three times, we get the sums smaller than $4d$, but repeating them four or more times, we get the sums equal to or greater than $4d$. Therefore from angles of squares or regular pentagons, only trihedral convex angles can be formed. The angles of regular hexagons are $\frac{4}{3}d$, and of regular polygons with more than 6 sides even greater. The sum of three or more of such angles will be equal to or greater than $4d$. Therefore no convex polyhedral angles can be formed from such angles.

It follows that only the following five types of regular polyhedra can occur: those whose faces are regular triangles, meeting by three, four or five triangles at each vertex, or those whose faces are either squares, or regular pentagons, meeting by three faces at each vertex.

Regular polyhedra of each of the five types do exist and are often called **Platonic solids** after the Greek philosopher Plato. They are:

(i) **regular tetrahedron** whose surface is formed by 4 regular triangles (Figure 79);

(ii) **octahedron** whose surface is formed by 8 regular triangles (Figure 80);

(iii) **icosahedron** whose surface is formed by 20 regular triangles (Figure 81);

(iv) cube (or **hexahedron**) whose surface consists of 6 squares (Figure 82);

(v) **dodecahedron** whose surface is formed by 12 regular pentagons (Figure 83).
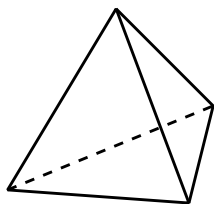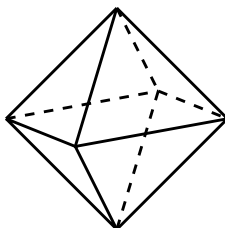
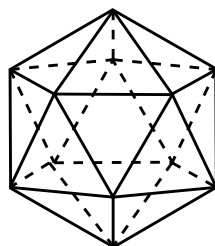Figure 79   Figure 80   Figure 81
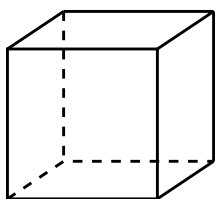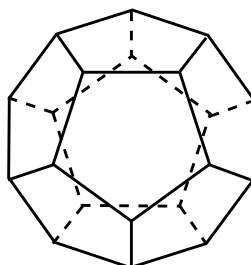
Figure 82   Figure 83

86. Constructions of Platonic solids. The above argument shows that regular polyhedra, if they exist, fall into five types, but it does not prove that regular polyhedra of each of the five types exist. In order to establish their existence it suffices to point out a construction of each of the five Platonic solids. In the case of the cube, which was defined as a rectangular parallelepiped all of whose